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Basic kinetic wealth-exchange models: common features and open problems

M. Patriarca[1,2,a], E. Heinsalu[1,2], and A. Chakraborti[3]

[1] IFISC, Instituto de Física Interdisciplinar y Sistemas Complejos (CSIC-UIB), 07122 Palma de Mallorca, Spain
[2] National Institute of Chemical Physics and Biophysics, Rävala 10, 15042 Tallinn, Estonia
[3] Laboratoire de Mathématiques Appliquées aux Systèmes, École Centrale Paris, 92290 Châtenay-Malabry, France

Abstract. We review the basic kinetic wealth-exchange models of Angle [J. Angle, Social Forces **65**, 293 (1986); J. Math. Sociol. **26**, 217 (2002)], Bennati [E. Bennati, Rivista Internazionale di Scienze Economiche e Commerciali **35**, 735 (1988)], Chakraborti and Chakrabarti [A. Chakraborti, B. K. Chakrabarti, Eur. Phys. J. B **17**, 167 (2000)], and of Dragulescu and Yakovenko [A. Dragulescu, V.M. Yakovenko, Eur. Phys. J. B **17**, 723 (2000)]. Analytical fitting forms for the equilibrium wealth distributions are proposed. The influence of heterogeneity is investigated, the appearance of the fat tail in the wealth distribution and the relaxation to equilibrium are discussed. A unified reformulation of the models considered is suggested.

PACS. 89.75.-k Complex systems – 89.65.Gh Economics; econophysics, financial markets, business and management – 02.50.-r Probability theory, stochastic processes, and statistics

1 Introduction

Many scientists have underlined the importance of a quantitative approach in social sciences [1–8]. In fact, statistical mechanics and social sciences have been always linked to each other in a constructive way, due to the statistical character of the objects of study [9,10]. On one hand, various discoveries, first made in the field of social sciences, introduced new concepts which turned out to be relevant for the development of statistical mechanics and later of the science of complex systems. For instance, fat tails were found by Pareto in the distribution of wealth [11,12]; the first description of financial time series through statistical mechanics, made by Bachelier in his PhD thesis [13–15], also represents the first formalization of a stochastic process in terms of the random walk model; large fluctuations were observed by Mandelbrot in the time series of cotton price [16]. On the other hand, physics has often represented a prototype for modelling economic systems. For example, many works of Paul Samuelson were inspired by thermodynamics; the analogies between physics and economics were studied by Jan Tinbergen in his PhD thesis entitled "Minimum Problems in Physics and Economics". Recent developments of economics rely more and more on the theory of stochastic processes and the science of complex systems [17].

The present paper considers some models of wealth exchange between individuals or economical entities, introduced independently in different fields such as social sciences, economics, and physics. We refer to them as *kinetic wealth-exchange models* (KWEM), since they provide a description of wealth flow in terms of stochastic wealth exchange between *agents*, resembling the energy transfer between the molecules of a fluid [2,18,19]. In order to maintain the discussion at a fundamental level, we limit ourselves to the following simple KWEMs: those introduced by Angle (A-models) [20–23], Bennati (B-model) [24–26], Chakraborti and Chakrabarti (C-model) [27], and by Dragulescu and Yakovenko (D-model) [28]. The goal of the paper is to discuss their general common features, formulation, and stationary solutions for the wealth distribution. We consider a heterogeneous KWEM, in order to illustrate how a simple KWEM can generate realistic wealth distributions. We also clarify some relevant issues, recently discussed in the literature, concerning the relaxation to equilibrium and the appearance of a power law tail of the equilibrium distribution in heterogeneous models.

A noteworthy difficulty in the study of wealth or money exchanges based on a kinetic approach had been pointed out by Mandelbrot [29]:

> ... there is a great temptation to consider the exchanges of money which occur in economic interaction as analogous to the exchanges of energy which occur in physical shocks between gas molecules... Unfortunately the Pareto distribution decreases much more slowly than any of the usual laws of physics...

[a] e-mail: marco.patriarca@gmail.com

The problem referred to in this quotation is that the asymptotic shape of the energy distributions of gases predicted by statistical mechanics usually have the Gibbs form or a form with an exponential tail. The real wealth distributions, instead, exhibit a Pareto power law tail [11,12,30–32],

$$f(x) \sim 1/x^{1+\alpha}, \tag{1}$$

with $1 < \alpha < 2$. However, it has become clear that (a) the actual shapes of wealth distribution at intermediate values of wealth are well fitted by a Γ- or an exponential distribution [21,33–35], so that they can be reproduced also by simple KWEMs with homogeneous agents (see Sect. 3); (b) KWEMs with suitably *diversified* agents can generate also the power law tail of the wealth distribution [2,19,36] (see Sect. 4). This has opened the way to a simple, quantitative approach in modelling real wealth distributions as arising from wealth exchanges among economical units.

The paper is structured as follows: in Section 2 a general description of a KWEM is given. In Section 3 the homogeneous A-, B-, C-, and D-models are discussed. Explicit analytical fitting forms for the equilibrium wealth distributions are given. In Section 4 we discuss the influence of heterogeneity, taking the heterogeneous C-model as a representative example. In this respect, we analyze the mechanism leading to a robust power law tail. Some issues concerning the convergence time scale of the model and the related finite cut-off of the power law are discussed. In Section 5 a unified reformulation of the A-, C-, and D-models is suggested, which in turn naturally lends itself to further generalizations. An example of generalized model is worked out in detail. Conclusions are drawn in Section 6.

2 General structure

In the models under consideration the system is assumed to be made up of N agents with wealths $\{x_i \geq 0\}$ ($i = 1, 2, \ldots, N$). At every iteration an agent j exchanges a quantity Δx with another agent k chosen randomly. The total wealth $X = \sum_i x_i$ is constant as well as the average wealth $\langle x \rangle = X/N$. After the exchange the new values x'_j and x'_k are ($x'_j, x'_k \geq 0$)

$$\begin{aligned} x'_j &= x_j - \Delta x, \\ x'_k &= x_k + \Delta x. \end{aligned} \tag{2}$$

Here, without loss of generality, the minus (plus) sign has been chosen in the equation for the agent j (k). The form of the function $\Delta x = \Delta x(x_j, x_k)$ defines the underlying dynamics of the model.

In KWEMs, agents can be characterized by an exchange parameter $\omega \in (0, 1]$ which defines the maximum fraction of the wealth x that enters the exchange process. Equivalently, one can introduce the saving parameter $\lambda = 1 - \omega$, with value in the interval $[0, 1)$, representing the minimum fraction of x preserved during the exchange. The parameter ω (λ) also determines the time

scale of the relaxation process as well as the mean value $\langle x \rangle$ at equilibrium [37]. If the value of ω (λ) is the same for all the agents, the model is referred to as *homogeneous* (see Sect. 3). If the agents assume different values ω_i (λ_i) then the model is called *heterogeneous* (see Sect. 4). Homogeneous models can reproduce the shape of the Γ-distribution observed in real data at small and intermediate values of the wealth. For $\omega < 1$ ($\lambda > 0$), they have the self-organizing property to converge toward a stable state with a wealth distribution which has a non-zero mode, differently from a purely exponential distribution. Models with suitably diversified agents can reproduce also the power law tail (1) found in real wealth distributions.

In actual economic systems the total wealth is not conserved and a more faithful description should be used. It is therefore interesting to observe how the closed economy models considered here, in which $\sum_i x_i$ is constant, provide realistic shapes of wealth distributions. This suggests that the main factor determining the wealth distribution is the *wealth exchange.*

When the variation of wealths is not due to an actual exchange between the two agents but the quantity Δx is entirely lost by one agent and gained by the other one, the model is called *unidirectional.* Furthermore, it is possible to conceive multi-agent interaction models, not considered here, in which a number $M > 2$ of agents enter each trade. Then the evolution law has the more general form $x'_i = x_i + \Delta x_i$, with $i = 1, \ldots, M$, $\sum_{i=1}^{M} \Delta x_i = 0$, and the Δx_i depending somehow on the wealths x_i of the M interacting agents.

3 Homogeneous models

3.1 A1-model

Here we consider the model introduced by Angle in 1983 in references [20,21], referred to as A1-model (a different model of Angle, the One-Parameter Inequality Process, referred to as A2-model, is consider in Sect. 3.2 below). The A-models are inspired by the surplus theory of social stratification and describe how a non-uniform wealth distribution arises from wealth exchanges between individuals.

The A1-model is unidirectional and its dynamics is highly nonlinear. The dynamical evolution is determined by equations (2) with Δx given as

$$\Delta x = \epsilon \omega \left[\eta_{j,k} x_j - (1 - \eta_{j,k}) x_k \right]. \tag{3}$$

Here ϵ and $\eta_{j,k}$ are random variables. The first one is a random number in the interval $(0, 1)$, which can be distributed either uniformly or with a certain probability distribution $g(\epsilon)$, as in some generalizations of the basic A1-model [20]. The second one is a random dichotomous variable responsible for the unidirectionality of the wealth flow as well as for the nonlinear character of the dynamics. It is a function of the difference between the wealths of the interacting agents j and k, $\eta_{j,k} \equiv \phi(x_k - x_j)$, assuming the value $\eta_{j,k} = 1$ with probability p_0 for $x_j > x_k$ or the
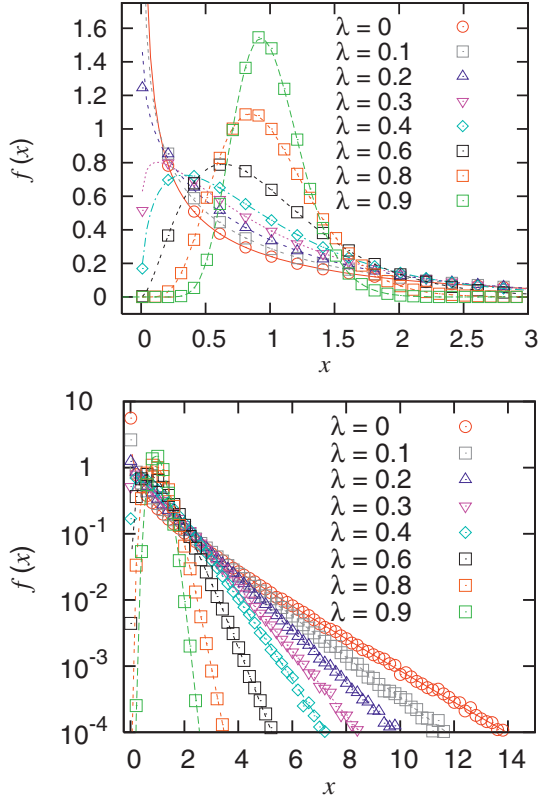
Fig. 1. (Color online) Equilibrium wealth distribution for the basic version of the A1-model defined by equations (3), for the case $p_0 = 1/2$: results of numerical simulations (symbols) and fitting functions equationqs (4) (curves) for different values of the saving parameter $\lambda = 1 - \omega$ in linear (above) and semi-log (below) scale. The value of n is governed by equation (5). In this simulation the average wealth is $\langle x \rangle = 1$.

value $\eta_{j,k} = 0$ with probability $1 - p_0$ for $x_k > x_j$. The value $\eta_{j,k} = 1$ produces a wealth transfer $|\Delta x| = \epsilon \omega x_j$ from agent j to k, while the value $\eta_{j,k} = 0$ corresponds to a wealth transfer $|\Delta x| = \epsilon \omega x_k$ from k to j.

A special case of the A1-model is that with symmetrical interaction, obtained for $p_0 = 1/2$. Notice that for this value of p_0 the random variable $\eta_{j,k} \equiv \eta$ becomes independent of x_j and x_k. We have studied this particular case through numerical simulations for various values of the saving parameter λ. The system considered was made up of $N = 10^5$ agents with equal initial wealths $x_{0i} = 1$ and the transactions were performed until equilibrium was reached. The equilibrium distributions in Figure 1 were obtained by averaging over 10^5 different runs. They are well fitted by the Γ-distribution

$$f(x) = \beta \gamma_n(\beta x) = \frac{\beta}{\Gamma(n)}(\beta x)^{n-1} \exp(-\beta x), \quad (4)$$

where

$$\beta^{-1} = \langle x \rangle / n, \quad (5)$$

$$n \equiv \frac{D}{2} = \frac{1 + 2\lambda}{2(1 - \lambda)} = \frac{3}{2\omega} - 1. \quad (6)$$

Since $\lambda = 1 - \omega \in [0, 1)$, the parameter n is a real number in the interval $[1/2, \infty)$. Notice that from equations (4) and (5) it follows that for $n < 1$, i.e., for $\lambda < 1/4$ ($\omega > 3/4$), the Γ-distribution diverges for $x \to 0$, as visible in Figure 1 for the cases $\lambda = 0$, $\lambda = 0.1$, and $\lambda = 0.2$. For the critical value $\lambda = 1/4$ ($\omega = 3/4$), separating the distributions which diverge from those which go to zero for $x \to 0$, an exponential distribution is obtained,

$$f(x) = \beta \gamma_1(\beta x) = \beta \exp(-\beta x). \quad (7)$$

The A1-model has a simple mechanical analogue if the quantity $D = 2n$ defined in equation (5) is interpreted as an effective dimension for the system and β^{-1} as a temperature. It is easy to check that the distribution $\gamma_n(\beta x) \equiv \gamma_{D/2}(\beta x)$ given by equation (4) is the equilibrium distribution for the kinetic energy of a perfect gas in D dimensions as well as for the potential energy of a D-dimensional harmonic oscillator or a general harmonic system with D degrees of freedom. This definition of effective dimension is consistent with the equipartition theorem, since

$$\langle x \rangle = n\beta^{-1} = D\beta^{-1}/2, \quad (8)$$

see reference [38] for details.

3.2 A2-model

The One-Parameter Inequality Process model, here referred to as A2, is another model introduced by Angle and is described in detail in references [22,23]. It differs from the A1-model considered above in that it only employs a stochastic dichotomic variable η_{jk}, which can assume randomly the values $\eta_{jk} = 0$ or $\eta_{jk} = 1$. The model is defined by equations (2) with

$$\Delta x = -\eta_{jk}\omega x_k + (1 - \eta_{jk})\omega x_j. \quad (9)$$

The model describes a unidirectional flow of wealth from agent k toward agent j for $\eta_{jk} = 1$ or vice versa for $\eta_{jk} = 0$. For the particular case in which the two values of η_{jk} are always equiprobable, one can rewrite the process, without loss of generality, with a $\Delta x = \omega x_k$ in equations (2). Numerical simulations of this model confirm the findings of references [22,23], that for small enough ω the stationary wealth distribution is well fitted by a Γ-distribution $\gamma_n(x)$, with $n \approx 1/\omega - 1 = \lambda/(1 - \lambda)$. We find that this fitting (not shown) is very good at least up to $\lambda \approx 0.7$.

3.3 B-model

Another KWEM was introduced in 1988 by Bennati [24, 25]. Its basic version, that we discuss here, is a simple unidirectional model where units exchange constant amounts Δx_0 of wealth [24–26]. In principle, in the B-model a situation where the wealths of the agents would become negative could occur. This is prevented allowing the transaction to take place only if the condition $x'_j, x'_k \geq 0$ is
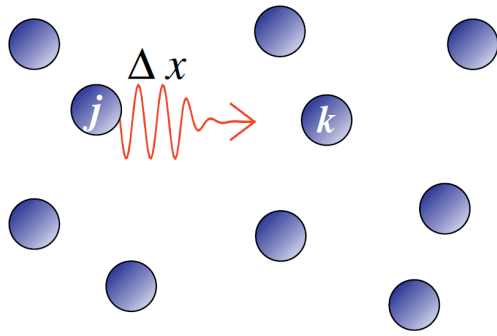
Fig. 2. (Color online) In the B-model the quantity x can only vary by a constant amount $\Delta x = \Delta x_0$, which can e.g. be lost by a unit j and then absorbed by a unit k, analogously to the emission-absorption process of light quanta of constant frequency.

fulfilled, i.e., the process is described by equation (2) with $\Delta x = \Delta x_0$ if $x'_j, x'_k \geq 0$ and with $\Delta x = 0$ otherwise. Since the wealth can vary only by a constant amount Δx_0, the model reminds a set of particles exchanging energy by emitting and re-absorbing light quanta, as illustrated symbolically in Figure 2. Analytically the equilibrium state of the B-model is well described by the exponential distribution (7). A main difference respect to the other models considered here is that in the B-model the amount of wealth exchanged between the two agents is independent of x_i, while in the other models represents a multiplicative random process, since $\Delta x \propto x_i$.

3.4 C-model

In the model introduced in 2000 by Chakraborti and Chakrabarti [27] the general exchange rule reads,

$$x'_j = \lambda x_j + \epsilon(1-\lambda)(x_j + x_k),$$
$$x'_k = \lambda x_k + \bar{\epsilon}(1-\lambda)(x_j + x_k), \qquad (10)$$

where $\bar{\epsilon} = 1 - \epsilon$. Here the new wealth x'_j (x'_k) is expressed as a sum of the saved fraction $\lambda x'_j$ ($\lambda x'_k$) of the initial wealth and a random fraction ϵ ($\bar{\epsilon}$) of the total remaining wealth, obtained summing the respective contributions of agents j and k. Equations (10) are equivalent to equations (2), with

$$\Delta x = \omega(\bar{\epsilon} x_j - \epsilon x_k) = (1-\lambda)(\bar{\epsilon} x_j - \epsilon x_k). \qquad (11)$$

Like in the A1-model, at equilibrium the system is well described by a Γ-distribution (4). For the parameter n we find now [38,39]

$$n \equiv \frac{D}{2} = \frac{1+2\lambda}{1-\lambda} = \frac{3}{\omega} - 2, \qquad (12)$$

which is twice the value of the corresponding parameter of the A1-model with $p_0 = 1/2$, discussed in Section 3.1.

In Figure 3 numerical results are compared with the fitting based on equation (12). In this case the probability density is always finite for $x \to 0$, since for $\lambda = 0$ ($\omega = 1$) one has $n = 1$ and the distribution does not diverge, being equal to the exponential function (7).
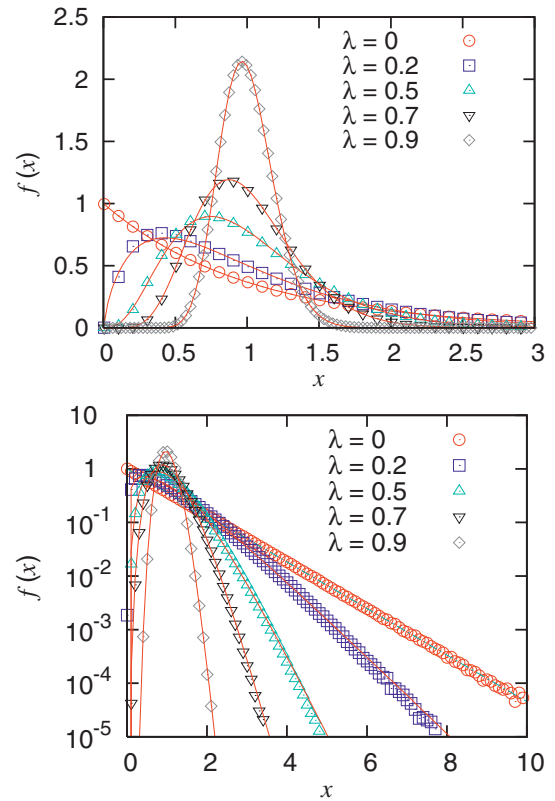

Fig. 3. (Color online) Equilibrium wealth distributions in linear and semi-log scale for different values of the saving parameter λ in the closed economy model defined by equations (10). The continuous curves are the fitting functions, i.e. a Γ-distribution of the form equation (4) just as for the A1-model, but the value of $n(\lambda)$ is given by equation (12).

3.5 D-model

The models introduced in 2000 by Dragulescu and Yakovenko [28] were conceived to describe flow and distribution of money. They have a sound interpretation both of the conservation law $x'_j + x'_k = x_j + x_k$, since money is measured in the same unit and conserved during transactions, and of the stochasticity of the update rule, representing a randomly chosen realization of trade. Various models were considered in reference [28], with a Δx either constant (similarly to the B-model discussed above) or dependent on the values x_i of the agents; also more realistic models, in which e.g. firms were introduced or debts were allowed. For simplicity, we consider among them the model which probably best represents the random character of KWEMs, referred to as the D-model below, in which the total initial amount $x_j + x_k$ is reshuffled randomly between the two interacting units,

$$x'_j = \epsilon(x_j + x_k),$$
$$x'_k = \bar{\epsilon}(x_j + x_k). \qquad (13)$$

Equivalently, the dynamics can be described by equations (2), with

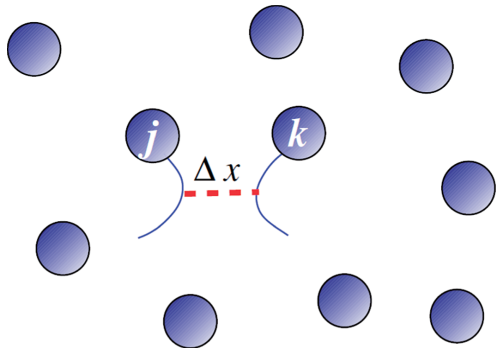$$\Delta x = \bar{\epsilon} x_j - \epsilon x_k. \qquad (14)$$

Fig. 4. (Color online) The D-model as well as the C-model prescribe a microscopic interaction between two units analogously to a kinetic model of gas in which, during an elastic collision, two generic particles j and k exchange an energy amount Δx.

Table 1. Comparison of dependence of the fitting parameter n in the Γ-distribution (4) on λ or ω, for the basic homogeneous versions of the A-, C-, and D-models.

Model	$n(\omega)$	$n(\lambda)$
A1	$3/2\omega - 1$	$(1 + 2\lambda)/2(1 - \lambda)$
A2	$1/\omega - 1$	$\lambda/(1 - \lambda)$
C	$3/\omega - 2$	$(1 + 2\lambda)/(1 - \lambda)$
D	1	1

The D-model is formally recovered from the C-model for $\lambda = 0$ ($\omega = 1$).

The equilibrium distribution of the D-model is well fitted by the exponential distribution (7). A mechanical analogue of the D-model is a gas, in which particles undergo pair collisions in which some energy is exchanged [40], as symbolically illustrated in Figure 4.

3.6 Stationary wealth distributions

The parameters of the Γ-distribution, obtained from the fitting of the wealth distributions of the stationary solutions for the models considered, are summarized in Table 1. The analytical forms of the respective parameters n, given as a function of ω or λ, provide a good fitting: for the model A2, the fitting is good only up to $\lambda \approx 0.7$. The close analogies among the various models are evident, however the existence of a general solution has not been demonstrated, see e.g. references [23,41].

4 Influence of heterogeneity

Here we discuss the influence of heterogeneity, considering as an example the generalization of the C-model. Heterogeneity is introduced by assigning a different parameter ω_i (λ_i) to each agent i. The formulation of the heterogeneous models can be straightforwardly obtained from those of the corresponding homogeneous ones by replacing the generic term ωx_i (λx_i) with $\omega_i x_i$ ($\lambda_i x_i$) in the evolution law. In the case of the C-model equations (10)

become

$$x'_j = \lambda_j x_j + \epsilon \left[(1 - \lambda_j) x_j + (1 - \lambda_k) x_k \right],$$
$$x'_k = \lambda_k x_k + \bar{\epsilon} \left[(1 - \lambda_j) x_j + (1 - \lambda_k) x_k \right], \qquad (15)$$

and the exchanged amount of wealth in equations (2) is now

$$\Delta x = \bar{\epsilon} \omega_j x_j - \epsilon \omega_k x_k = \bar{\epsilon} (1 - \lambda_j) x_j - \epsilon (1 - \lambda_k) x_k. \quad (16)$$

The set of parameters $\{\omega_i\}$ ($\{\lambda_i\}$) is constant in time and specifies the profiles of the agents. The values $\{\omega_i\}$ ($\{\lambda_i\}$) are assumed to be distributed in the interval between 0 and 1 with probabilities h_i (g_i) and $\sum_i h_i = 1$ ($\sum_i g_i = 1$). In the limit of an infinite number of agents, one can introduce a probability distribution $h(\omega)$ [$g(\lambda)$], with $\int_0^1 d\omega h(\omega) = 1$ [$\int_0^1 d\lambda g(\lambda) = 1$].

Various analytical and numerical studies of this model have been carried out [2,19,22,36,41–48] and as a main result it has been found that the exponential law remains limited to intermediate x-values, while a Pareto power law appears at larger values of x. Such a shape is prototypical for real wealth distributions. Numerical simulations and theoretical considerations suggest that the power law exponent is quite insensitive to the details of the system parameters, i.e., to the distribution $h(\omega)$. In fact, the Pareto exponent depends on the limit $g(\lambda \to 1)$. If $g(\lambda) \sim (1 - \lambda)^{\alpha-1}$ with $\lambda \to 1$ and $\alpha \leq 1$, then the corresponding power law has an exponent α [44]. Thus, in general, agents with λ_i close to 1 are responsible for the appearance of the power law tail [44,47,49].

Probably the most interesting feature of the equilibrium state is that while the shape of the wealth distribution $f_i(x)$ of agent i is a Γ-distribution, the sum of the wealth distributions of the single agents, $f(x) = \sum_i f_i(x)$, produces a power law tail. Vice versa, one could say that the global wealth distribution $f(x)$ can be resolved as a mixture of partial wealth probability densities $f_i(x)$ with exponential tail, with different parameters. For instance, the corresponding average wealth depends on the saving parameter as $\langle x \rangle_i \propto 1/(1 - \lambda_i) = 1/\omega_i$; see references [47,49,50] for details.

Importantly, all real distributions have a finite cutoff; no real wealth distribution has an infinitely extended power law tail. The Pareto law is always observed between a minimum wealth value $x_{\min}$ and a cutoff $x_{\max}$, representing the wealth of the richest agent. This can be well reproduced by the heterogeneous model using an upper cutoff $\lambda_{\max} < 1$ for the saving parameter distribution $g(\lambda)$: the closer to one is $\lambda_{\max}$, the larger is $x_{\max}$ and wider the interval in which the power law is observed [49].

The role of the λ-cutoff is closely related to and relevant for understanding the relaxation process. The relaxation time scales of single agents in a heterogeneous model are proportional to $1/(1 - \lambda_i)$ [37]. This means that the slowest convergence rate is determined by $1 - \lambda_{\max}$. In numerical simulations of heterogeneous KWEMs, one necessarily employes a finite λ-cutoff. However, this should not be regarded as a limit of numerical simulations but a feature suited to describe real wealth distributions. Simulations confirm the fast convergence to equilibrium for each

agent with the above mentioned time scale [37]. Gupta has demonstrated numerically that the convergence is exponentially fast [51].

In reference [52] it has been claimed that heterogeneous KWEMs with randomly distributed λ_i ($0 \leq \lambda_i < 1$) cannot undergo a fast relaxation toward an equilibrium wealth distribution, but the relaxation should instead take place on algebraic time scale. This in turn means that there cannot exists any power law tail. Such claims are probably correct for systems with a λ-distribution $g(\lambda)$ rigorously extending as far as $\lambda = 1$, corresponding to a power law tail extending as far as $x = \infty$. However, this does not apply to KWEMs with a saving parameter cutoff $\lambda_{\max} < 1$, which is the natural choice in describing real systems, as well as in numerical simulations, employing a finite λ-cutoff: in this case the largest time scale is finite and relaxation is fast.

5 Generalizations

In this section a unified reformulation of the exchange laws of the A-, C-, and D-models is suggested and as an example an application to the C-model is made.

5.1 Reformulation

It is possible to reformulate the evolution law either through a single stochastic saving variable $\tilde{\lambda}$ or an equivalent stochastic exchange variable $\tilde{\omega} = 1 - \tilde{\lambda}$. This formal rearrangement of the equations maintains the form of the evolution law very simple and has at the same time the advantage to be particularly suitable to make further generalizations. For the sake of generality, we consider the case of a heterogeneous system characterized by a parameter set $\{\omega_i\}$. The models discussed above (apart from the B-model) can be rewritten according to the basic equations (2), where the wealth exchange term is now given by

$$\Delta x = \tilde{\omega}_j x_j - \tilde{\omega}_k x_k. \tag{17}$$

The meaning of the new stochastic variables $\tilde{\omega}_j$ and $\tilde{\omega}_k$ introduced is simple: $\tilde{\omega}_j$ represents the fraction of wealth given by agent j to k during the transaction, and vice versa for $\tilde{\omega}_k$. Comparison with the equations defining the A-, C-, and D-models provides the following definitions for $\tilde{\omega}_j$ and $\tilde{\omega}_k$:

– In the A1-model, $\tilde{\omega}_j$ and $\tilde{\omega}_k$ are independent nonlinear stochastic functions of the agent wealths x_j and x_k,

$$\begin{aligned} \tilde{\omega}_j &= \eta_{j,k} \epsilon \omega_j, \\ \tilde{\omega}_k &= (1 - \eta_{j,k}) \epsilon \omega_k, \end{aligned} \tag{18}$$

where $\eta_{j,k} = \phi(x_k - x_j) = 1$ with probability p_0 for $x_k - x_j > 0$ and $\eta_{j,k} = 0$ with probability $1 - p_0$ for $x_k - x_j < 0$, while ϵ is a random number in $(0,1)$. For $\eta_{j,k} = 0$ one has $\tilde{\omega}_j = 0$ and $\tilde{\omega}_k \in (0, \omega_k)$, whereas for $\eta_{j,k} = 1$ one has $\tilde{\omega}_k = 0$ and $\tilde{\omega}_j \in (0, \omega_j)$.

Table 2. Comparison of the A-, C-, and D-models: explicit forms of the random variables $\tilde{\omega}_j$ and $\tilde{\omega}_k$ in the unified reformulation (17). See text for details.

Model	$\tilde{\omega}_j$	$\tilde{\omega}_k$
A1	$\epsilon \, \eta \, \omega_j$	$(1 - \eta) \, \epsilon \, \omega_k$
A2	$\eta \, \omega_j$	$(1 - \eta) \, \omega_k$
C	$\epsilon \, \omega_j$	$(1 - \epsilon) \, \omega_k$
D	ϵ	$(1 - \epsilon)$

– In the A2-model, $\tilde{\omega}_j$ and $\tilde{\omega}_k$ only contain the dichotomic variable,

$$\begin{aligned} \tilde{\omega}_j &= \eta_{j,k} \omega_j, \\ \tilde{\omega}_k &= (1 - \eta_{j,k}) \, \omega_k. \end{aligned} \tag{19}$$

– For the C-model,

$$\begin{aligned} \tilde{\omega}_j &= \epsilon \omega_j, \quad \tilde{\omega}_j \in (0, \omega_j), \\ \tilde{\omega}_k &= (1 - \epsilon) \omega_k, \quad \tilde{\omega}_k \in (0, \omega_k), \end{aligned} \tag{20}$$

where ϵ is a random number in $(0, 1)$.
– The D-model is recovered from equations (20) of the C-model when $\omega_i = 1$ for each agent i.

The reformulation is summarized in Table 2 with reference to equation (17). Different generalizations can now be done changing only the properties of the stochastic variables $\tilde{\omega}_i$, while maintaining the same formulation (17) of the exchange law.

5.2 An example

As an example which can be represented through equation (17), we consider a generalization of the homogeneous C-model. In the original version there is a constraint on the maximum fraction of invested wealth, given by a value $0 < \omega \leq 1$ of the exchange parameter, or equivalently on the minimum saved fraction, given by a value $0 \leq \lambda < 1$ of the saving parameter. Now an additional constraint on the minimum fraction of the invested wealth is assumed. This may describe e.g. trades which always have a minimum risk for an agent. It can be represented by an analogous parameter ω', with $0 < \omega' < \omega$, representing the minimum fraction of wealth invested in a single trade. One can also define a parameter $\lambda' = 1 - \omega'$, with $\lambda < \lambda' < 1$, representing the maximum fraction of saved wealth (i.e. it is not possible to go through a trade without losing a non-zero amount of wealth). Then the stochastic variables $\tilde{\omega}_i$ in equation (17) become uniform random numbers in intervals defined by the parameters ω' and ω (or by λ' and λ),

$$\tilde{\omega}_i \in (\omega', \omega) = (1 - \lambda', 1 - \lambda). \tag{21}$$

We have performed numerical simulations for a set of combinations of parameters (λ, λ') and found that the equilibrium distributions are always well fitted by the same Γ-distribution (5). However, we have not found a simple analytical formula for fitting the dependence of the parameter n on the saving parameters λ and λ'. The behavior of
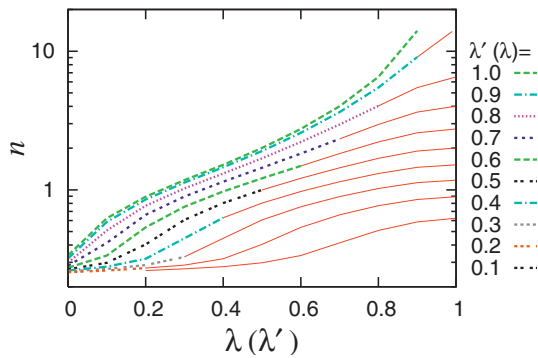
Fig. 5. (Color online) Fitting parameter $n(\lambda, \lambda')$ of the Γ-distribution $\gamma_n(x)$ for the generalized model with $\lambda < \tilde{\lambda}_i < \lambda'$. Dotted and dashed lines: n as a function of λ ($\leq \lambda'$ by definition) for fixed values of λ' shown on the right side. These lines stop at $\lambda = \lambda'$ since by definition $\lambda \leq \lambda'$. Continuous lines: n as a function of λ' ($\geq \lambda$) for the same fixed values of λ shown on the right side.

n versus λ (λ') is represented graphically in Figure 5. Dotted/dashed curves (different colors) represent n versus λ for the different fixed values of λ' shown on the right side These curves stop at $\lambda = \lambda'$, since by definition $\lambda < \lambda'$. From there the continuous (red) curves start, which represent n versus λ' for the same fixed values of λ listed in the legend on the right. The first (dashed green) curve from the top extending on the whole interval $\lambda = (0, 1)$ represents n as a function of λ f or $\lambda' = 1$ and corresponds to the original homogeneous C-model. For this particular case $n(\lambda)$ is known to diverge as $n \sim 1/(1 - \lambda) \sim 1/\omega$ for $\lambda \to 1$ (see Tab. 1), while in all the other cases n is finite.

6 Conclusions and discussion

We have reviewed some basic KWEMs of closed economy systems, introduced by scientists working in different fields, allowing us to point out analogies and differences between them. We have first considered the homogeneous models and then discussed the influence of heterogeneity. The heterogeneous KWEMs are particularly relevant in the study of real wealth distributions, since they can reproduce both the exponential shape at intermediate values of wealth as well as the power law tail.

In all the models discussed, including the heterogeneous one, the equilibrium wealth distribution of a single agent is well fitted by a Γ-distribution, known to be the canonical distribution of a general harmonic system with a suitable number of degrees of freedom. This suggests a simple mechanism underlying the (approach to) equilibrium of these systems, similar to the energy redistribution in a mechanical system. However, a general demonstration that the Γ-distribution is the stationary solution of KWEMs and an understanding of how it arises is still missing (see Refs. [23,41,53,54] for theoretical considerations on and the microeconomic formulation of this issue).

Furthermore, we have discussed how in a heterogeneous KWEM the sum of the single agent wealth distributions can produce a power law tail. In particular, we have clarified some issues concerning the relaxation process and the existence of power law tails: whenever there is a finite cutoff in the saving parameter distribution, the largest time scale of the system is finite and one observes a fast (exponential) relaxation toward a power law, which extends over a finite interval of wealth. The width of such interval depends on the saving parameter cutoff.

Due to the similarity of the structures of the models discussed, we have proposed a novel unified reformulation based on the introduction of suitable stochastic variables $\tilde{\omega}_i$, representing the *actual fraction of wealth lost by the i-th agent* during a single transaction. This unified formulation lends itself easily to further generalizations, which can be obtained by modifying the stochastic properties of the variable $\tilde{\omega}_i$ only, while leaving the general evolution law unchanged. We have illustrated the new formulation by working out in detail an example, in which the fraction of wealth lost $\tilde{\omega}_i$ is characterized by a lower as well as an upper limit.

Besides the KWEMs considered in the present paper, originally formulated through finite time difference stochastic equations, other relevant (versions of) KWEMs have been introduced in the literature; see references [10,19,55] for an overview. Their mathematical formulation can be similar to the one of the present paper [21–23,36,56], or different approaches can be used, such as matrix theory [57], the master equation [35,58,59], the Boltzmann equation [41,52,60–63], the Lotka-Volterra equation [64,65], or Markov chains models [66–68]. All these models share a description of wealth flow as due to exchanges between basic units. In this respect, they are all very different from the class of models formulated in terms of a Langevin equation for a single wealth variable subjected to multiplicative noise [29,69–72]. The latter models can lead to wealth distributions with a power law tail. In fact, they converge toward a log-normal distribution, which, however, does not fit real wealth distributions as well as a Γ-distribution or a β-distribution and is asymptotically characterized by too large variances [21].

Finally, we would like to point out that even though KWEMs have been the subject of intensive investigations, their economical interpretation is still an open problem. It is important to keep in mind that in the framework of a KWEM the agents should not be related to the *rational agents* of neoclassical economics: an interaction between two agents does not represent the effect of decisions taken by two economic agents who have full information about the market and behave rationally in order to maximize their utility. The description of wealth flow provided by KWEMs takes into account the stochastic element, which does not respond by definition to any rational criterion. Also some terms employed in the study of KWEMs, such as *saving propensity* (replaced here by *saving parameter*), *risk aversion*, etc., can be misleading since they seem to imply a decisional aspect behind the behavior of agents. Trying to interpret the dynamics of KWEMs through concepts taken from the neoclassical theory leads to obvious misunderstandings [73]. However, it is interesting to note that very recently, Chakrabarti and Chakrabarti have put

forward a microeconomic formulation of the above models, using the utility function as a guide to the behavior of agents in the economy [54]. Instead, KWEMs provide a description at a coarse grained level, as in the case of many statistical mechanical models, where the connection with the microscopic mechanisms is not visible; however, the equivalence is maintained.

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