Opinion formation in kinetic exchange models: Spontaneous symmetry-breaking transition

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We propose a minimal multiagent model for the collective dynamics of opinion formation in the society by modifying kinetic exchange dynamics studied in the context of income, money, or wealth distributions in a society. This model has an intriguing spontaneous symmetry-breaking transition to polarized opinion state starting from nonpolarized opinion state. In order to analyze the model, we introduce an iterative map version of the model, which has very similar statistical characteristics. An approximate theoretical analysis of the numerical results is also given, based on the iterative map version.

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I. INTRODUCTION

Recently physicists have been studying social phenomena and dynamics leading to the growth of the interdisciplinary field of "sociophysics" [1]. One of the problems is of opinion formation, which is a collective dynamical phenomenon, and as such is closely related to the problems of competing cultures or languages [2,3]. It deals with a measurable response of the society to, e.g., political issues, acceptances of innovations, etc. Numerous models of competing options have been introduced to study this phenomenon, e.g., the voter model (which has a binary opinion variable with the opinion alignment proceeding by a random choice of neighbors) [4], or the Sznajd-Weron discrete opinion formation model (where more than just a pair of spins is associated with the decision making procedure) [5]. There have been other studies of systems with more than just two possible opinions [6] or where the opinion of individuals is represented by a continuous variable [7–9] using real numbers. Also, since opinion formation in a human society is mediated by social interactions between individuals, such social dynamics has been considered to take place on a network of relationships (see [2] for recent review on such models).

A two-body exchange dynamics has already been developed in the context of modeling income, money, or wealth distributions in a society [10–14]. The general aim was to study a many-agent statistical model of closed economy (analogous to the kinetic theory model of ideal gases) [15], where N agents exchange a quantity x that may be defined as wealth. The states of agents are characterized by the wealth $\{x_i\}$, $i=1,2,\ldots,N$, such that $x_i>0$, $\forall i$ and the total wealth $W=\Sigma_i x_i$ is conserved. The question of interest is "what is the equilibrium distribution of wealth $f(x)$, such that $f(x)dx$ is the probability that in the steady state of the system a randomly chosen agent will be found to have wealth between x and $x+dx$?" The evolution of the system is carried out according to a prescription, which defines the trading rule between agents. The agents interact with each other through a pairwise interaction characterized by a "saving" parameter λ, with $0\leq\lambda\leq 1$. The dynamics of the model introduced by Chakraborti and Chakrabarti (CC) is as follows [15]:

$$x_i' = \lambda x_i + \epsilon(1-\lambda)(x_i + x_j),$$

$$x_j' = \lambda x_j + (1-\epsilon)(1-\lambda)(x_i + x_j), \quad (1)$$

where ϵ $(0\leq\epsilon\leq 1)$ is a stochastic variable, changing with time. It can be noticed that in this way, the quantity x is conserved during the single transactions: $x_i'+x_j'=x_i+x_j$, where x_i' and x_j' are the agent wealth after the transaction has taken place. In general, the functional form for steady-state distribution $f(x)$ is seen to be close to the Γ distribution [16,17]. As a further generalization, the agents could have different saving propensities, and the steady-state distribution $f(x)$ shows Pareto-like power-law behavior asymptotically [18,19].

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Earlier, Toscani [20] introduced and discussed kinetic models of (continuous) opinion formation involving both exchange of opinion between individual agents and diffusion of information. Based on this model, During *et al.* [21] proposed another mathematical model for opinion formation in a society that is made of two groups: one group of ordinary people and one group of strong opinion leaders. Starting from microscopic interactions among individuals, they arrived at a macroscopic description of the opinion formation process. They discussed the steady states of the system and extended it to incorporate emergence and decline of opinion leaders. Here, we report the studies of a minimal model for the collective dynamics of opinion formation in the society, based on kinetic exchanges.

II. MODEL FOR OPINION FORMATION AND RESULTS

A. Homogeneous multiagent model

Following CC's model described in the earlier section, we present a minimal model [22] for the collective dynamics of opinion $O_i(t)$ of the ith person in the society, consisting of N ($N\to\infty$) persons. We assume that any particular person can discuss (interact) only with one other person each time (time increases discretely by unity after each such discussion). A two-person "discussion" is viewed here as a simple two-body *scattering process* in physics. Persons in the society may bump onto each other randomly and exchange opinions through such random two-person discussions. In general, a person i could have any opinion O_i between two extreme polarities denoted by $+1$ and -1. In any discussion at time $t+1$, a person *retains* a fraction of his or her older opinion $O_i(t)$, determined by his or her "conviction," parametrized by λ_i. This parameter value is characteristic of a person and does not change with time t. Additionally, the person i is "influenced" stochastically by the other person j during the discussion having the influence parameter equal to his or her conviction parameter λ_j. We further assume for simplicity that all agents are *homogeneous*—have the *same* conviction parameter λ. Mathematically the dynamics may be represented by

$$O_i(t+1) = \lambda[O_i(t) + \epsilon_t O_j(t)],$$

$$O_j(t+1) = \lambda[O_j(t) + \epsilon_t' O_i(t)], \tag{2}$$

where the opinion $-1 \le O_i(t) \le 1$ for all agents i and time t, the conviction parameter $0 \le \lambda \le 1$ is *quenched* (does not change with time), and the stochastic parameters ϵ_t and ϵ_t' are *annealed* variables (change with time)—uncorrelated random numbers uniformly distributed between zero and unity. Note that the equations are linear, but nonlinearity is introduced in this model by imposing that $-1 \le O_i(t) \le 1$ for all agents i and times t.

The question we are interested in is that if such social dynamics continually take place, can any consensus be reached or can polarity evolve after a long time? Mathematically, we are interested in the steady-state distribution of O and other statistical properties. It is noteworthy that unlike in the market models, here we have no conservation of opinion.

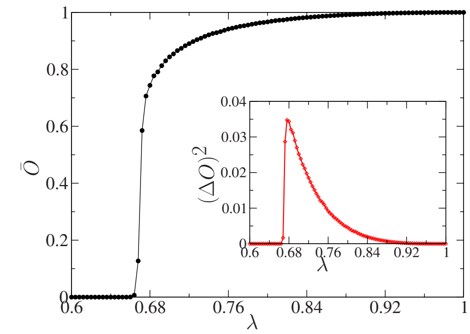

FIG. 1. (Color online) Numerical results for the variation of the average opinion $\bar{O}(t)$ for large t (steady-state value of $\bar{O}$) against λ, following dynamics of Eq. (2). (Inset) Numerical results for the variation of the variance $(\Delta O)^2 \equiv (O-\bar{O})^2$ against λ, following dynamics of Eq. (2).

Rather, the steady state of value of $\bar{O}(t)=(1/N)|\Sigma_i O_i(t)|$ represents the order of the average opinion in the society after a long time t. We study the relaxation dynamics in the society: the relaxation and fluctuation of $\bar{O}$ and the steady-state value of $\bar{O}(t)$ for $t>\tau$, the relaxation time.

Remarkably, we find that there is an appearance of polarity or consensus, starting from initial random disorder (where O_i's are uniformly distributed with positive and negative values). In the language of physics, there is a "spontaneous symmetry-breaking" transition in the system: starting from $\bar{O}(0)=0$ the system evolves either to the "para" state with $\bar{O}\equiv\bar{O}(t>\tau)=0$ (where all individual agents have the opinion 0) for $\lambda \le 2/3$ or (*continuously*) to the "symmetry broken" state $\bar{O}\equiv\bar{O}(t>\tau)\neq 0$ (where *all* individuals have either positive or negative opinions) for $\lambda \ge 2/3$ (see Fig. 1) for times $t>\tau$. We note, however, that the fluctuation in $\bar{O}$ does not diverge and shows a cusp near λ_c (see the inset of Fig. 1). We also study the relaxation behavior of $\bar{O}(t)$ and the critical divergence of the relaxation time τ near $\lambda=\lambda_c=2/3$ (see Sec. II C; Fig. 4).

B. Random multiplier map

The basic nature of transition produced by Eq. (2) can also be found in the following simple iterative map:

$$O(t+1) = \lambda(1+\epsilon_t)O(t), \tag{3}$$

with the restriction that $O(t) \le 1$, which is ensured by assuming that if $O(t) \ge 1$, $O(t)$ is set equal to 1. As usual, ϵ_t is a stochastic variable ranging between 0 and 1 (assumed to be uniformly distributed in our case). In a mean-field approach, the above equation reduces effectively to a multiplier map like $O(t+1)=\lambda(1+\langle\epsilon\rangle)O(t)$, where $\langle\epsilon\rangle=1/2$. Clearly for $\lambda \le 2/3$, $O(t)$ converges to zero. The initial value $O(0)$ is assigned either a positive or negative value. If it starts from a positive (negative) value, $O(t)$ remains positive (negative). We note that there are subtle differences in the dynamics of Eqs. (2) and (3). Apart from the absence of spontaneous symmetry breaking of the multiagent model [from $\pm O_i(0)$

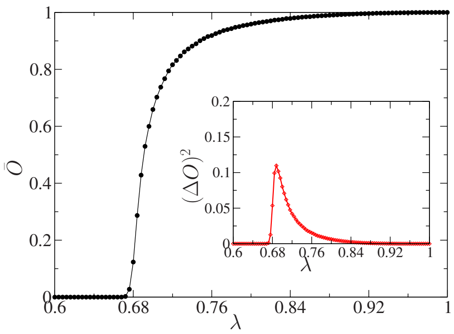

FIG. 2. (Color online) Numerical results for the variation of the average opinion $\bar{O}(t)$ for large t (steady-state value of $\bar{O}$) against λ, following dynamics of Eq. (3). (Inset) Numerical results for the variation of the variance $(\Delta O)^2 \equiv (O-\bar{O})^2$ against λ, following dynamics of Eq. (3).

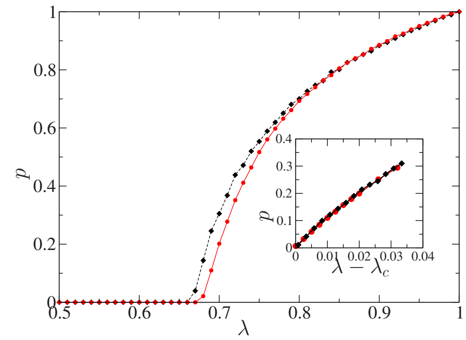

FIG. 3. (Color online) Numerical results for the variation of the average condensate fraction $p(t)$ for large t (steady-state value of p) against λ, following dynamics of Eq. (2) in black diamonds and dynamics of Eq. (3) in red circles. (Inset) Numerical results for the growth of p, following dynamics of Eq. (2) in black diamonds and dynamics of Eq. (3) in red circles, close to λ_c.

values to all positive or all negative transitions beyond λ_c], the nature of the phase transition (singularity) in the iterative map is also slightly different. The critical value $\lambda_c = \exp\{-(2\ln 2 - 1)\} \approx 0.6796$ has an analytical derivation [23], but for most numerical studies done here, we take $\lambda_c = 0.68$. The time variation of the average opinion $\bar{O}(t) = (1/N)\Sigma_i |O_i(t)|$, where i refers to different initial realizations and N refers to the total of all such realizations, and its fluctuations are studied numerically (see Fig. 2). We study the relaxation behavior of $\bar{O}(t)$ and the critical divergence of the relaxation time τ near $\lambda = \lambda_c = 0.68$ (see Sec. II C; Fig. 5). We note again that the fluctuation in $\bar{O}$ does not diverge and shows a cusp near $\lambda_c = 0.68$ (see the inset of Fig. 2). We also note that the steady-state fluctuation ΔO near λ_c is generally much higher in magnitude for the map case.

C. Results and analyses

For both the multiagent model and the iterative map, we study the variation (with λ) of the fraction p of the agents having $O_i = \pm 1$ at any time t in the steady state ($t > \tau$). This parameter p gives the average "condensation" fraction (of people in the society having extreme opinions $|O_i| = 1$) in the steady state. The growth of p, as shown in Fig. 3, is seen to be similar to that of $\bar{O}$. The inset shows that the growth behaviors for p above (respective) λ_c for both the multiagent model and map are identical.

We studied the relaxation behaviors of $\bar{O}$ and p for both the multiagent model and map. In each case, the relaxation time is estimated numerically from the time value at which $\bar{O}$ or p first touches the steady-state value within a preassigned error limit. We find diverging growth of relaxation time τ (for both $\bar{O}$ and p) near $\lambda = \lambda_c$ (see Figs. 4 and 5). The values of exponent z for the divergence in $\tau \sim |\lambda - \lambda_c|^{-z}$ have been estimated numerically for both the multiagent model and the map (for both $\lambda > \lambda_c$ and $\lambda < \lambda_c$, wherever accurate data were obtained). For the multiagent model, the fitting values for exponent z corresponding to $\bar{O}$ and p, respectively, are $z \approx 1.0 \pm 0.1$ and $z \approx 0.7 \pm 0.1$. For the map case, the fitting values for exponent z corresponding to both $\bar{O}$ and p turn out to be the same: $z \approx 1.5 \pm 0.1$.

For the iterative map (3), we study carefully the time evolution of the condensation fraction p of $|O| = 1$ in different realizations at different values of λ. The variation of the steady-state value p against λ is shown in Fig. 2. It may be noted that while the steady-state value of $\bar{O}$ starts to grow from $\lambda \approx 2/3$ (see Fig. 2), the steady-state value of p starts growing at $\lambda \approx 0.68$ (see Fig. 3). Numerical results for the growth of the relaxation time τ for both $\bar{O}$ and p against λ are shown in Fig. 5. Both diverge at $\lambda \approx 0.68$. This clearly indicates that p, rather than $\bar{O}$, is the order parameter for the transition.

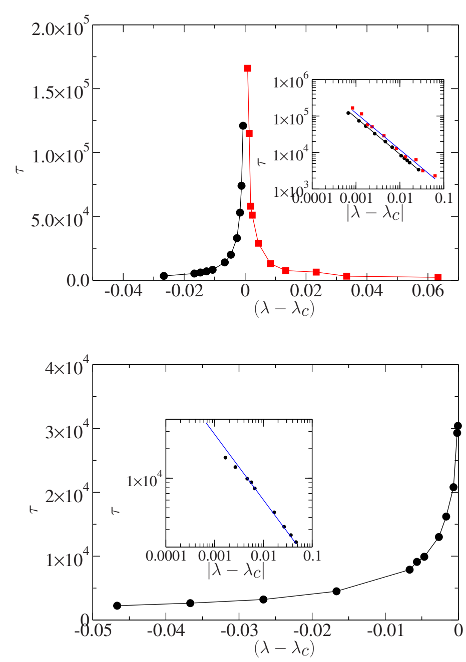

FIG. 4. (Color online) Numerical results for relaxation time behaviors τ versus $\lambda - \lambda_c$ for (top) multiagent model with $\bar{O}$ and (bottom) multiagent model with p. (Insets) Determination of exponent z from numerical fits of $\tau \sim |\lambda - \lambda_c|^{-z}$.

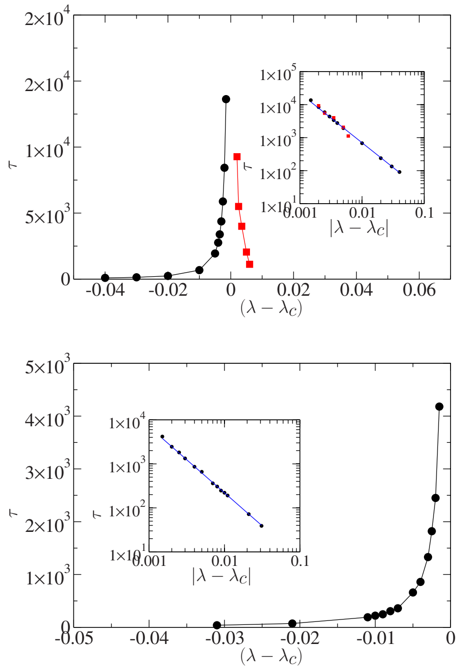

FIG. 5. (Color online) Numerical results for relaxation time behaviors τ versus $\lambda-\lambda_c$, for (top) map with $\bar{O}$ and (bottom) map with p. (Insets) Determination of exponent z from numerical fits of $\tau \sim |\lambda-\lambda_c|^{-z}$.

An approximate analysis of the above transition for λ closer to unity can be done for the iterative map Eq. (3) as follows. In Fig. 6, we give the numerical results for the steady-state distribution opinion $P(|O|)$ for three different values of λ; we observe roughly a bimodal nature of the distribution as $\lambda \to 1$: one mode is the uniform distribution within the range $|O_{min}| < |O| < 1$ (and $|O_{min}| \approx \lambda$) and another a δ function at $|O|=1$. We therefore approximate the steady-state distribution of opinion by assuming that opinion $O(t)$ is distributed uniformly starting from a minimum O_{min} up to unity with (integrated) probability $(1-p)$ and a δ function at exactly unity with probability p. Then,

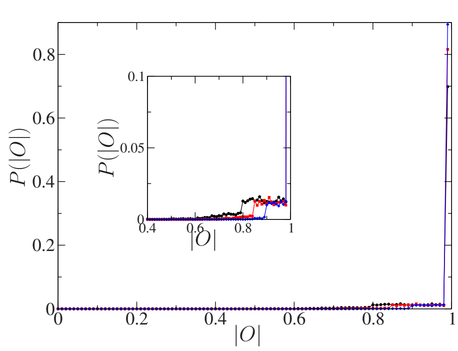

FIG. 6. (Color online) Numerical results for the steady-state distribution opinion $P(|O|)$ for three values $\lambda=0.8, 0.85, 0.9$ showing bimodal distributions in each case. (Inset) The same steady-state distribution $P(|O|)$ for three values $\lambda=0.8, 0.85, 0.9$, but close to $|O|=1$. In the main panel and in the inset, the black circles, red squares, and blue diamonds represent values of $\lambda=0.8, 0.85$, and 0.9, respectively.

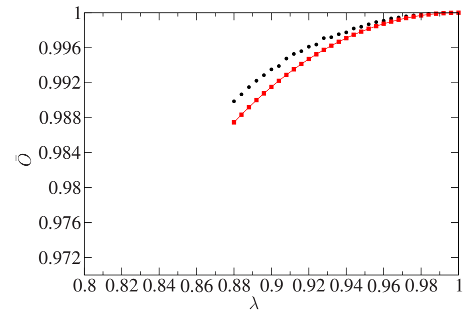

FIG. 7. (Color online) Fit of the approximate theoretical calculation (5) (in red squares) with the numerical simulations for $\lambda \to 1$, following dynamics of Eq. (3) (in black circles).

$$\bar{O} = (1-p)O_{av} + p \times 1, \quad (4)$$

where $O_{av}=(O_{min}+1)/2$. We have assumed that the value $O(t)$ stays in those two regions (from λ to 1 and exactly at 1) with probabilities $(1-p)$ and p. Hence, the corresponding equations are

$$O(t+1) = \lambda(1+\epsilon)O(t), \quad \text{with probability } 1-p,$$

$$O(t+1) = 1, \quad \text{with probability } p.$$

Note that the first equation is realized only if $\lambda(1+\epsilon)O(t) < 1$ or $\epsilon < \epsilon_{max} = (1/\lambda O_{av}) - 1$. This cutoff implies that $(1-p) = \int_0^{\epsilon_{max}} d\epsilon = (1/\lambda O_{av}) - 1$ since $\epsilon \sim \text{uni}[0,1]$. By substituting O_{av} and p in Eq. (4), we derive the result that

$$\bar{O} = \frac{5\lambda + 2\lambda^2 - \lambda^3 - 2}{2\lambda(1+\lambda)}, \quad (5)$$

which is compared with the numerical simulations for $\lambda \to 1$ in Fig. 7. It is evident that the approximation holds well, only for $\lambda \to 1$.

III. SUMMARY AND DISCUSSION

In summary, we proposed a minimal model for the collective dynamics of opinion formation in the society by modifying kinetic exchange dynamics studied in the context of markets. The multiagent model [dynamics given by Eq. (2)] and its map version [dynamics given by Eq. (3)] have kinetic exchange like linear contributions from random two-person discussions or scattering processes, although the saturation of $|O_i| \le 1$ induces nonlinearity in the dynamics. This model has an intriguing spontaneous symmetry-breaking transition to polarized opinion state starting from nonpolarized opinion state. Specifically, in the multiagent model, we see that for $\lambda > \lambda_c = 2/3$, starting from random positive and negative O_i values (or for that matter any arbitrary state), at $t=0$, the system eventually evolves to a state at $t > \tau$ where all O_i's are either positive or negative, with $|\bar{O}|$ determined by the λ value. This is similar to the growth of spontaneous magnetization in Ising magnets (where starting from arbitrary up- and down-spin states, a preferred direction is chosen by fluctuation), with magnetization determined by the temperature below its transition value. The appearance of spontaneous symmetry breaking in this simple kinetic opinion exchange model is truly remarkable. It appears to be one of the simplest collective dynamical models of many-body dynamics showing nontrivial phase-transition behavior. Indeed, it may be noted that for $\lambda \leq \lambda_c$, at $t > \tau$, all O_i's become identically zero (without any fluctuation), while for $\lambda > \lambda_c$, O_i's have fluctuations but the average has a steady-state value depending on the value of λ. We have only one absorbing state in our model and, as such, the nature of the phase transition in this model is quite different and does not fit to the commonly studied two absorbing state models (see, e.g., [24,25]).

We note that the model proposed by Hegselmann and Krause [7] is only similar to this model in the sense that opinions take real values in an interval. In their model, an agent i, with opinion O_i, interacts with neighboring agents whose opinions lie in the range $]O_i - \epsilon, O_i + \epsilon[$, where ϵ is the *uncertainty*, i.e., the agent i does not interact with a randomly chosen neighbor, but with all of its compatible neighbors at once. The Hegselmann-Krause update rule is intended to describe formal meetings, where there is an effective interaction involving many people at the same time (and not just random pairs like in our exchange model), such that the agent i takes the average opinion of its compatible neighbors. The model is fully determined by the uncertainty parameter ϵ; in our model, such a role is played by the conviction parameter λ.

In order to understand the nature of the transition, we also studied a simple iterative map and derived approximate result for the order-parameter variation under certain limits, which compares quite well with the numerical simulations. Specifically, we find that the fraction p of people with extreme opinion $|O_i| = 1$, and its fluctuations determine the nature of the phase transition in our model and locate the critical point accurately (from numerical studies). With the two-mode distribution (uniform and δ) of O, valid close to $\lambda \to 1$ (see Fig. 6), we could develop an approximate analysis of the variation of the steady-state mean opinion $|\bar{O}|$ against λ as in Eq. (5). In any case, further investigations are necessary for understanding this phase transition. Additional studies for the *heterogeneous* conviction factors λ_i in influence of "field terms" that represent the external influence of media, etc. will be reported elsewhere [26]. Also, the study of this phase-transition behavior for an extended model with separate conviction and influence parameters in Eq. (2) has recently been reported [27].

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