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Econophysics review: II. Agent-based models

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This article is the second part of a review of recent empirical and theoretical developments usually grouped under the heading *Econophysics*. In the first part, we reviewed the statistical properties of financial time series, the statistics exhibited in order books and discussed some studies of correlations of asset prices and returns. This second part deals with models in Econophysics from the point of view of agent-based modeling. Of the large number of multi-agent-based models, we have identified three representative areas. First, using previous work originally presented in the fields of behavioral finance and market microstructure theory, econophysicists have developed agent-based models of order-driven markets that we discuss extensively here. Second, kinetic theory models designed to explain certain empirical facts concerning wealth distribution are reviewed. Third, we briefly summarize game theory models by reviewing the now classic minority game and related problems.

Keywords: Econophysics; Financial time series; Correlation; Agent based modelling

1. Introduction

In the first part of the review, empirical developments in Econophysics were studied. We pointed out that some of these widely known 'stylized facts' are already at the heart of financial models. But many facts, especially the newer statistical properties of order books, have not yet been taken into account. As advocated by many during the financial crisis of 2007–2008 (see, e.g., Bouchaud (2008), Farmer and Foley (2009) and Lux and Westerhoff (2009)), agent-based models should play a large role in future financial modeling. In economic models, there is usually the representative agent, who is 'perfectly rational' and uses the 'utility maximization' principle while taking action. In contrast, multi-agent models, which originated from statistical physics considerations, allow us to go beyond the prototype theories with the 'representative' agent in traditional economics. In this second part of our review, we present recent developments concerning agent-based models in Econophysics.

There are, of course, many reviews and books already published in this area (see, e.g., Challet *et al.* (2004), Coolen (2005), Chatterjee and Chakrabarti (2007), Samanidou *et al.* (2007), Bouchaud *et al.* (2009), Lux and Westerhoff (2009), Yakovenko and Rosser (2009), etc.). We present here our perspectives in three representative areas.

2. Agent-based modeling of order books

2.1. Introduction

Although known, at least partly, for a long time—Mandelbrot (1963) presented a reference for a paper dealing with the non-normality of price time series in 1915, followed by several others in the 1920s—'stylized facts' have often been left aside when modeling financial markets. They were even often referred to as 'anomalous' characteristics, as if observations failed to comply with theory. Much has been done these past 15 years in order

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to address this challenge and provide new models that can reproduce these facts. These recent developments have been built on top of early attempts at modeling the mechanisms of financial markets with agents. For example, Stigler (1964), investigating rules of the SEC,† and Garman (1976), investigating the double-auction microstructure, are two of those historical works. It seems that the first modern attempts at this type of model were made in the field of behavioral finance. This field aims at improving financial modeling based on the psychology and sociology of the investors. Models are built with agents who can exchange shares of stocks according to exogenously defined utility functions reflecting their preferences and risk aversion. LeBaron (2006b) shows that this type of modeling offers good flexibility when reproducing some of the stylized facts, and LeBaron (2006a) provides a review of this type of model. However, although achieving some of their goals, these models suffer from many drawbacks: first, they are very complex, and it may be a very difficult task to identify the roles of their numerous parameters and the types of dependence on these parameters; second, the chosen utility functions do not necessarily reflect what is observed in the mechanisms of a financial market.

A sensible change in modeling is obtained with much simpler models implementing only well-identified and presumably realistic 'behavior'. Cont and Bouchaud (2000) use noise traders who are subject to 'herding', i.e. form random clusters of traders sharing the same view on the market. This idea is also used by Raberto *et al.* (2001). A complementary approach is to characterize traders as fundamentalists, chartists or noise traders. Lux and Marchesi (2000) propose an agent-based model in which these types of traders interact. In all these models, the price variation directly results from the excess demand: at each time step, all agents submit orders and the resulting price is computed. Therefore, everything is cleared at each time step and there is no order book structure to keep track of orders.

One big step is made with models taking into account limit orders and keeping them in an order book once submitted and not executed. Chiarella and Iori (2002) build an agent-based model where all traders submit orders depending on the three elements identified by Lux and Marchesi (2000): chartists, fundamentalists, noise traders. Orders submitted are then stored in a persistent order book. In fact, one of the first simple models with this feature was proposed by Bak *et al.* (1997). In this model, orders are particles moving along a price line, and each collision is a transaction. Due to the numerous caveats in this model, the authors propose in the same paper an extension with fundamentalists and noise traders in the spirit of the models previously evoked. Maslov (2000) goes further in the modeling of trading mechanisms by taking into account fixed limit orders and market orders that trigger transactions, and simulating the order book. This model was solved analytically by Slanina (2001) using a mean-field approximation.

Following this modeling trend, the more or less 'rational' agents composing models in economics tend to vanish and be replaced by the notion of flows: orders are no longer submitted by an agent following strategic behavior, but are viewed as an arriving flow, the properties of which are to be determined by empirical observations of market mechanisms. Thus, the modeling of order books calls for more 'stylized facts', i.e. empirical properties that could be observed on a large number of order-driven markets. Biais *et al.* (1995) provide a thorough empirical study of the order flows in the Paris Bourse a few years after its complete computerization. Market orders, limit orders, time of arrivals and placement are studied. Bouchaud *et al.* (2002) and Potters and Bouchaud (2003) provide statistical features of the order book itself. These empirical studies, which were reviewed in the first part of this review, are the foundation of 'zero-intelligence' models, in which 'stylized facts' are expected to be reproduced by the properties of the order flows and the structure of the order book itself, without considering exogenous 'rationality'. Challet and Stinchcombe (2001) propose a simple model of order flows: limit orders are deposited in the order book and can be removed if not executed, in a simple deposition–evaporation process. Bouchaud *et al.* (2002) use this type of model with an empirical distribution as input. As of today, the most complete empirical model is, to our knowledge, that of Mike and Farmer (2008), where order placement and cancelation models are proposed and fitted to empirical data. Finally, new challenges arise as scientists attempt to identify the simple mechanisms that allow an agent-based model to reproduce non-trivial behavior: herding behavior (Cont and Bouchaud 2000), dynamic price placement (Preis *et al.* 2007), threshold behavior (Cont 2007), etc.

In this part we review some of these models. This survey is, of course, far from exhaustive, and we have selected models that we feel are representative of a specific modeling trend.

2.2. *Early order-driven market modeling: Market microstructure and policy issues*

The pioneering works concerning the simulation of financial markets were aimed at studying market regulations. The very first one (Stigler 1964) attempted to investigate the effect of the regulations of the SEC on American stock markets using empirical data from the 1920s and 1950s. Twenty years later, at the start of the computerization of financial markets, Hakansson *et al.* (1985) implemented a simulator in order to test the feasibility of automated market making. Instead of reviewing the huge microstructure literature, we refer the reader to the well-known books of O'Hara (1995) and Hasbrouck (2007), for example, for a panoramic view of

†Security Exchange Commission.

this branch of finance. However, by presenting a small selection of early models, we underline the grounding of recent order book modeling.

2.2.1. A pioneer order book model. To our knowledge, the first attempt to simulate a financial market was by Stigler (1964). This paper was a biting and controversial reaction to the Report of the Special Study of the Securities Markets of the SEC (Cohen 1963a), the aim of which was to "study the adequacy of rules of the exchange and that the New York stock exchange undertakes to regulate its members in all of their activities" (Cohen 1963b). According to Stigler, this SEC report lacks rigorous tests when investigating the effects of regulation on financial markets. Stating that "demand and supply are [$\cdots$] erratic flows with sequences of bids and asks dependent upon the random circumstances of individual traders", he proposes a simple simulation model to investigate the evolution of the market. In this model, constrained by the simulation capability of 1964, price is constrained within $L = 10$ ticks. (Limit) orders are randomly drawn, in trade time, as follows: they can be bid or ask orders with equal probability, and their price level is uniformly distributed on the price grid. Each time an order crosses the opposite best quote, it is a market order. All orders are of size one. Orders not executed $N = 25$ time steps after their submission are canceled. Thus, N is the maximum number of orders available in the order book.

In the original paper, a run of a hundred trades was computed manually using tables of random numbers. Of course, no particular result concerning the 'stylized facts' of financial time series was expected at that time. However, in his review of order book models, Slanina (2008) performs simulations of a similar model, with parameters $L = 5000$ and $N = 5000$, and shows that price returns are not Gaussian: their distribution exhibits power-law behavior with exponent 0.3, far from empirical data. As expected, the limitation L is responsible for the sharp cut-off of the tails of this distribution.

2.2.2. Microstructure of the double auction. Garman (1976) provides an early study of the double auction market with a point of view that does not ignore the temporal structure, and defines order flows. Price is discrete and constrained to be within $\{p_1, p_L\}$. Buy and sell orders are assumed to be submitted according to two Poisson processes of intensities λ and μ. Each time an order crosses the best opposite quote, it is a market order. All quantities are assumed to be equal to one. The aim of the author was to provide an empirical study of the market microstructure. The main result of the Poisson model was to support the idea that the negative correlation of consecutive price changes is linked to the microstructure of the double auction exchange. This paper is very interesting because it can be seen as a precursor that clearly sets the challenges of order book modeling. First, the mathematical formulation is promising. With its fixed constrained prices, Garman (1976) can define the state of the order book at a given time as the vector $(n_i)_{i=1,\ldots,L}$ of awaiting orders (negative quantity for bid orders, positive for ask orders). Future analytical models will use similar vector formulations that can be cast into known mathematical processes in order to extract analytical results (see, e.g., Cont *et al.* (2008), reviewed below). Second, the author points out that, although the Poisson model is simple, the analytical solution is hard to work out, and he provides Monte Carlo simulations. The need for numerical and empirical developments is a constant in all following models. Third, the structural question is clearly asked in the conclusion of the paper: "Does the auction-market model imply the characteristic leptokurtosis seen in empirical security price changes?". The computerization of markets that was about to take place when this research was published—Toronto's CATS† opened a year later in 1977—motivated many following papers on the subject. As an example, let us cite Hakansson *et al.* (1985), who built a model to choose the right mechanism for setting clearing prices in a multi-securities market.

2.2.3. Zero intelligence. In the models of Stigler (1964) and Garman (1976), orders are submitted in a purely random way on the grid of possible prices. Traders do not observe the market and do not act according to a given strategy. Thus, these two contributions clearly belong to the class of 'zero-intelligence' models. To our knowledge, Gode and Sunder (1993) were the first to introduce the expression 'zero intelligence' in order to describe non-strategic behavior of traders. It is applied to traders that submit random orders in a double auction market. The expression has since been widely used in agent-based modeling, sometimes with a slightly different meaning (see more recent models described in this review). Gode and Sunder (1993) study two types of zero-intelligence traders. The first are unconstrained zero-intelligence traders. These agents can submit random orders at random prices, within the allowed price range $\{1, \ldots, L\}$. The second are constrained zero-intelligence traders. These agents also submit random orders, but with the constraint that they cannot cross their given reference price p_i^{R}: constrained zero-intelligence traders are not allowed to buy or sell at a loss. The aim of the authors was to show that double auction markets exhibit an intrinsic 'allocative efficiency' (the ratio between the total profit earned by the traders divided by the maximum possible profit) even with zero-intelligence traders. An interesting fact is that, in this experiment, price series resulting from actions by zero-intelligence traders are much more volatile than those obtained with constrained traders. This fact will be confirmed in future models where 'fundamentalists' traders, having a reference price, are expected to stabilize the market (see Lux and

†Computer Assisted Trading System.

Marchesi (2000) and Wyart and Bouchaud (2007) below). Note that the results were criticized by Cliff and Bruten (1997), who show that the observed convergence of the simulated price towards the theoretical equilibrium price may be an artefact of the model. More precisely, the choice of traders' demands carries a lot of constraints that alone explain the observed results.

Modern works in Econophysics owe a lot to these early models or contributions. Starting in the mid-1990s, physicists have proposed simple order book models directly inspired from Physics, where the analogy 'order ≡ particle' is emphasized. Three main contributions are presented in the next section.

2.3. *Order-driven market modeling in Econophysics*

2.3.1. The order book as a reaction–diffusion model. A very simple model taken directly from Physics was presented by Bak *et al*. (1997). The authors consider a market with N noise traders able to exchange one share of stock at a time. Price $p(t)$ at time t is constrained to be an integer (i.e. the price is quoted in the number of ticks) with an upper bound $\bar{p}$: $\forall t$, $p(t) \in \{0, \dots, \bar{p}\}$. Simulation is initiated at time 0 with half of the agents asking for one share of stock (buy orders, bid) with price

$$p_{\mathrm{b}}^{j}(0) \in \{0, \bar{p}/2\}, \quad j = 1, \dots, N/2, \tag{1}$$

and the other half offering one share of stock (sell orders, ask) with price

$$p_{\mathrm{s}}^{j}(0) \in \{\bar{p}/2, \bar{p}\}, \quad j = 1, \dots, N/2. \tag{2}$$

At each time step t, agents revise their offer by exactly one tick, with equal probability of going up or down. Therefore, at time t, each seller (respectively buyer) agent chooses his new price as

$$p_{\mathrm{s}}^{j}(t+1) = p_{\mathrm{s}}^{j}(t) \pm 1 \quad (\text{respectively } p_{\mathrm{b}}^{j}(t+1) = p_{\mathrm{b}}^{j}(t) \pm 1). \tag{3}$$

A transaction occurs when there exists $(i, j) \in \{1, \dots, N/2\}^2$ such that $p_{\mathrm{b}}^{i}(t+1) = p_{\mathrm{s}}^{j}(t+1)$. In such a case the orders are removed and the transaction price is recorded as the new price $p(t)$. Once a transaction has been recorded, two orders are placed at the extreme positions on the grid: $p_{\mathrm{b}}^{i}(t+1) = 0$ and $p_{\mathrm{s}}^{j}(t+1) = \bar{p}$. As a consequence, the number of orders in the order book remains constant and equal to the number of agents. Figure 1 shows an illustration of these moving particles.

As pointed out by the authors, this process of simulation is similar to the reaction–diffusion model $A + B \to \emptyset$ in Physics. In such a model, two types of particles are inserted on each side of a pipe of length $\bar{p}$ and move randomly with steps of size one. Each time two particles collide, they are annihilated and two new particles are inserted. The analogy is summarized in table 1. Following this analogy, it can thus be shown that the variation $\Delta p(t)$

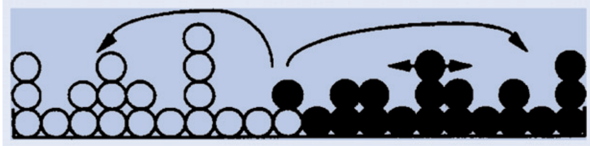

Figure 1. Illustration of the Bak, Paczuski and Shubik model: white particles (buy orders, bid) moving from the left, black particles (sell orders, ask) moving from the right. Reproduced from Bak *et al*. (1997).

Table 1. Analogy between the $A + B \to \emptyset$ reaction model and the order book of Bak *et al*. (1997).

Physics	Bak *et al*. (1997)
Particles	Orders
Finite pipe	Order book
Collision	Transaction

of the price $p(t)$ follows

$$\Delta p(t) \sim t^{1/4} \left(\ln\left(\frac{t}{t_0} \right) \right)^{1/2}. \tag{4}$$

Thus, on long time scales, the series of price increments simulated in this model exhibit a Hurst exponent $H = 1/4$. As for the stylized fact $H \approx 0.7$, this sub-diffusive behavior appears to be a step in the wrong direction compared with the random walk $H = 1/2$. Moreover, Slanina (2008) points out that no fat tails are observed in the distribution of the returns of the model, but rather it fits the empirical distribution with an exponential decay. Other drawbacks of the model can be mentioned. For example, the reintroduction of orders at each end of the pipe leads to an unrealistic shape of the order book, as shown in figure 2. Actually, this is the main drawback of the model: 'moving' orders is highly unrealistic as for modeling an order book, and since it does not reproduce any known financial exchange mechanism, it cannot be the basis for any larger model. Therefore, attempts by the authors to build several extensions of this simple framework, in order to reproduce 'stylized facts' by adding fundamental traders, strategies, trends, etc. are not of interest to us in this review. However, we feel that the basic model as such is very interesting because of its simplicity and its 'particle' representation of an order-driven market that has opened the way for more realistic models.

2.3.2. Introducing market orders. Maslov (2000) keeps the zero-intelligence structure of the (Bak *et al*. 1997) model, but adds more realistic features to order placement and evolution of the market. First, limit orders are submitted and stored in the model, without moving. Second, limit orders are submitted around the best quotes. Third, market orders are submitted to trigger transactions. More precisely, at each time step, a trader is chosen to perform an action. This trader can either submit a limit order with probability q_l or submit a

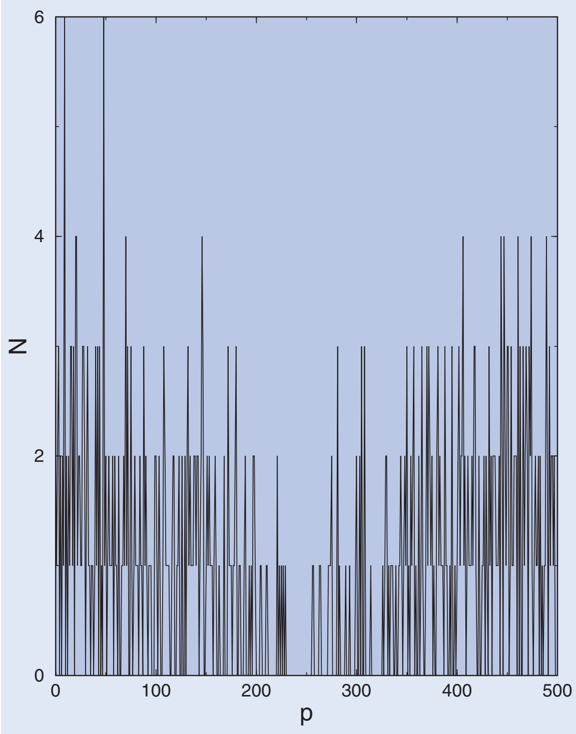

Figure 2. Snapshot of the limit order book in the Bak, Paczuski and Shubik model. Reproduced from Bak *et al.* (1997).

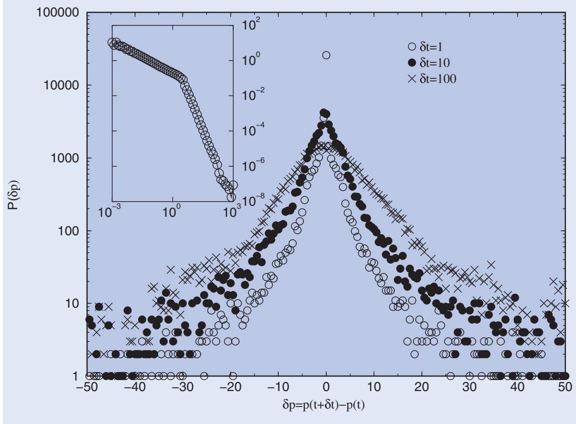

Figure 3. Empirical probability density functions of the price increments in the Maslov model. Inset: log–log plot of the positive increments. Reproduced from Maslov (2000).

Table 2. Analogy between the deposition–evaporation process and the order book of Challet and Stinchcombe (2001).

Physics	Challet and Stinchcombe (2001)
Particles	Orders
Infinite lattice	Order book
Deposition	Limit orders submission
Evaporation	Limit orders cancelation
Annihilation	Transaction

market order with probability $1-q_l$. Once this choice is made, the order is a buy or sell order with equal probability. All orders have a one-unit volume.

As usual, we denote $p(t)$ as the current price. In the case where the submitted order at time step $t+1$ is a limit ask (respectively bid) order, it is placed in the book at price $p(t)+\Delta$ (respectively $p(t)-\Delta$), Δ being a random variable uniformly distributed in $]0;\Delta^M=4]$. In the case where the submitted order at time step $t+1$ is a market order, one order at the opposite best quote is removed and the price $p(t+1)$ is recorded. In order to prevent the number of orders in the order book from increasing significantly, two mechanisms are proposed by the author: either keeping a fixed maximum number of orders (by discarding new limit orders when this maximum is reached), or removing them after a fixed lifetime if they have not been executed. Numerical simulations show that this model exhibits non-Gaussian heavy-tailed distributions of returns. Figure 3 plots the empirical probability density of the price increments for several time scales.

For a time scale $\delta t=1$, the author fit the tails distribution with a power law with exponent 3.0, i.e. this is reasonable compared with the empirical value. However, the Hurst exponent of the price series is still $H=1/4$ in this model. It should also be noted that Slanina (2001) proposed an analytical study of the model using a mean-field approximation (see section 2.5). This model introduces very interesting innovations in order book simulation: an order book with (fixed) limit orders, market orders, and the necessity of canceling orders waiting too long in the order book. These features are of prime importance in any following order book model.

2.3.3. The order book as a deposition–evaporation process. Challet and Stinchcombe (2001) continued the work of Bak *et al.* (1997) and Maslov (2000) and developed the analogy between the dynamics of an order book and an infinite one-dimensional grid, where particles of two types (ask and bid) are subject to three types of events: *deposition* (limit orders), *annihilation* (market orders) and *evaporation* (cancelation). Note that annihilation occurs when a particle is deposited on a site occupied by a particle of another type. The analogy is summarized in table 2.

Hence, the model proceeds as follows. At each time step, a bid (respectively ask) order is deposited with probability λ at a price $n(t)$ drawn according to a Gaussian distribution centred on the best ask $a(t)$ (respectively best bid $b(t)$) and with variance depending linearly on the spread $s(t)=a(t)-b(t)$: $\sigma(t)=K\,s(t)+C$. If $n(t)>a(t)$ (respectively $n(t)<b(t)$), then it is a market order: annihilation takes place and the price is recorded. Otherwise, it is a limit order and it is stored in the book. Finally, each limit order stored in the book has a probability δ of being canceled (evaporation). Figure 4 shows the average return as a function of the time scale. It appears that the series of price returns simulated with this

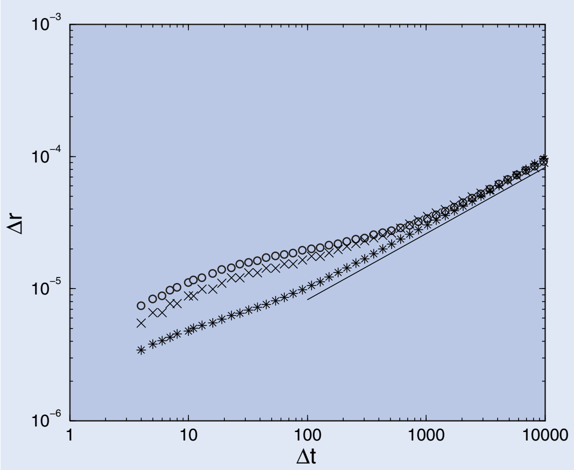

Figure 4. Average return $\langle r_{\Delta t}\rangle$ as a function of Δt for different sets of parameters and simultaneous depositions allowed in the Challet and Stinchcombe model. Reproduced from Challet and Stinchcombe (2001).

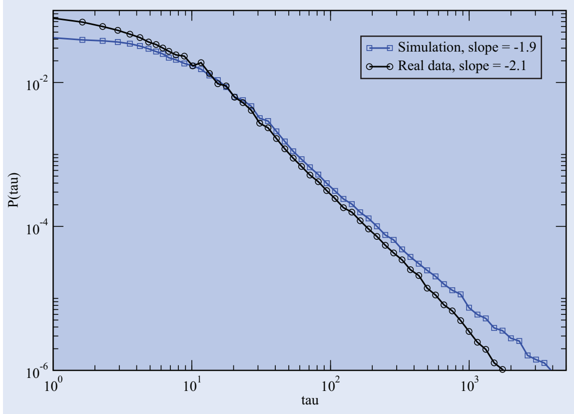

Figure 5. Lifetime of orders for simulated data in the Mike and Farmer model compared with the empirical data used for fitting. Reproduced from Mike and Farmer (2008).

model exhibit a Hurst exponent $H=1/4$ for short time scales, and that tends to $H=1/2$ for larger time scales. This behavior might be the consequence of the random evaporation process (which was not modelled by Maslov (2000), where $H=1/4$ for large time scales). Although some modifications of the process (more than one order per time step) seem to shorten the sub-diffusive region, it is clear that no over-diffusive behavior is observed.

2.4. *Empirical zero-intelligence models*

The three models presented in section 2.3 have successively isolated the essential mechanisms that are to be used when simulating a 'realistic' market: one order is the smallest entity of the model; the submission of one order is the time dimension (i.e. event time is used, not an exogenous time defined by market clearing and 'tatonnement' on exogenous supply and demand functions); submission of market orders (such as in Maslov (2000), as 'crossing limit orders' as in Challet and Stinchcombe (2001)); and cancelation of orders is taken into account. On the one hand, one may try to describe these mechanisms using a small number of parameters, using a Poisson process with constant rates for order flows, constant volumes, etc. This might lead to analytically tractable models, as will be described in section 2.5. On the other hand, one may try to fit more complex empirical distributions to market data without analytical concern.

This type of modeling is best represented by Mike and Farmer (2008). It is the first model that proposes an advanced calibration on the market data as for order placement and cancelation methods. As for volume and time of arrivals, the assumptions of previous models still hold: all orders have the same volume, and discrete event time is used for simulation, i.e. one order (limit or market) is submitted per time step. Following Challet and Stinchcombe (2001), there is no distinction between market and limit orders, i.e. market orders are limit orders that are submitted across the spread $s(t)$. More precisely, at each time step, one trading order is simulated: an ask (respectively bid) trading order is randomly placed at $n(t)=a(t)+\delta a$ (respectively $n(t)=b(t)+\delta b$) according to a Student distribution with scale and degrees of freedom calibrated on market data. If an ask (respectively bid) order satisfies $\delta a<-s(t)=b(t)-a(t)$ (respectively $\delta b>s(t)=a(t)-b(t)$), then it is a buy (respectively sell) market order and a transaction occurs at price $a(t)$ (respectively $b(t)$.

During a time step, several cancelations of orders may occur. The authors propose an empirical distribution for cancelation based on three components for a given order.

- The position in the order book, measured as the ratio $y(t)=\Delta(t)/\Delta(0)$, where $\Delta(t)$ is the distance of the order from the opposite best quote at time t.
- The order book imbalance, measured by the indicator $N_{\text{imb}}(t)=N_{\text{a}}(t)/(N_{\text{a}}(t)+N_{\text{b}}(t))$ (respectively $N_{\text{imb}}(t)=N_{\text{b}}(t)/(N_{\text{a}}(t)+N_{\text{b}}(t))$) for ask (respectively bid) orders, where $N_{\text{a}}(t)$ and $N_{\text{b}}(t)$ are the number of orders at the ask and bid in the book at time t.
- The total number $N(t)=N_{\text{a}}(t)+N_{\text{b}}(t)$ of orders in the book.

Their empirical study led them to assume that the cancelation probability has an exponential dependence on $y(t)$, a linear dependency on N_{imb} and finally decreases approximately as $1/N_t(t)$ as for the total number of orders. Thus, the probability $P(C\mid y(t), N_{\text{imb}}(t), N_t(t))$ to cancel an ask order at time t is formally written as

$$P(C\mid y(t),N_{\text{imb}}(t),N_t(t))=A(1-\mathrm{e}^{-y(t)})(N_{\text{imb}}(t)+B)\frac{1}{N_t(t)}, \tag{5}$$

where the constants A and B are to be fitted to market data. Figure 5 shows that this empirical formula provides quite a good fit to market data. Finally, the authors mimic the observed long memory of order signs by

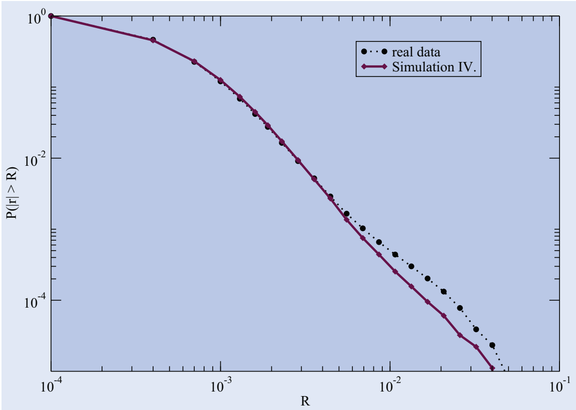

Figure 6. Cumulative distribution of returns in the Mike and Farmer model compared with the empirical data used for fitting. Reproduced from Mike and Farmer (2008).

simulating a fractional Brownian motion. The autocovariance function $\Gamma(t)$ of the increments of such a process exhibits a slow decay:

$$\Gamma(k) \sim H(2H-1)t^{2H-2}, \tag{6}$$

and it is therefore easy to reproduce exponent β of the decay of the empirical autocorrelation function of order signs observed on the market with $H = 1 - \beta/2$. The results of this empirical model are quite satisfying with respect to the return and spread distribution. The distribution of returns exhibits fat tails that are in agreement with empirical data, as shown in figure 6. The spread distribution is also very well reproduced. As their empirical model was constructed on the data of only one stock, the authors test their model on 24 other data sets of stocks in the same market and find, for half of them, good agreement between empirical and simulated properties. However, the bad results of the other half suggest that such a model is still far from being 'universal'.

Despite these very nice results, some drawbacks have to be pointed out. The first is the fact that the stability of the simulated order book is far from ensured. Simulations using empirical parameters may produce situations where the order book is emptied by large consecutive market orders. Thus, the authors require that there is at least two orders on each side of the book. This exogenous trick might be important, since it is activated precisely in the case of rare events that influence the tails of the distributions. Also, the original model does not focus on volatility clustering. Gu and Zhou (2009) propose a variant that tackles this feature. Another important drawback of the model is the way order signs are simulated. As noted by the authors, using an exogenous fractional Brownian motion leads to correlated price returns, which is in contradiction with empirical stylized facts. We also find that, on long time scales, it leads to a dramatic increase in the volatility. As we have seen in the first part of this review, the correlation of trade signs can, at least partly, be seen as an artefact of execution strategies. Therefore, this element is one of the numerous factors that should be taken into account when 'programming' the agents of the model. In order to do so, we have to leave the (quasi) 'zero-intelligence' world and see how modeling based on heterogeneous agents might help to reproduce non-trivial behavior. Prior to this development, discussed in section 2.6, we briefly review some analytical studies on 'zero-intelligence' models.

2.5. Analytical treatment of zero-intelligence models

In this section we present analytical results obtained for zero-intelligence models where processes are kept sufficiently simple so that a mean-field approximation may be derived (Slanina 2001) or probabilities conditional on the state of the order book may be computed (Cont *et al.* 2008). The key assumptions here are such that the process describing the order book is stationary. This allows us either to write a stable density equation, or to fit the model in a nice mathematical framework such as ergodic Markov chains.

2.5.1. Mean-field theory. Slanina (2001) proposes an analytical treatment of the model introduced by Maslov (2000) and reviewed above. Let us briefly described the formalism used. The main hypothesis is the following: on each side of the current price level, the density of limit orders is uniform and constant (ρ_+ on the ask side, ρ_- on the bid side). In that sense, this is a 'mean-field' approximation since the individual position of a limit order is not taken into account. Assuming we are in a stable state, the arrival of a market order of size s on the ask (respectively bid) side will make the price change by $x_+ = s/\rho_+$ (respectively $x_- = s/\rho_-$). It is then observed that the transformations of the vector $X = (x_+, x_-)$ occurring at each event (new limit order, new buy market order, new sell market order) are linear transformations that can easily and explicitly be written. Therefore, an equation satisfied by the probability distribution P of the vector X of price changes can be obtained. Finally, assuming further simplifications (such as $\rho_+ = \rho_-$), one can solve this equation for a tail exponent and find that the distribution behaves as $P(x) \approx x^{-2}$ for large x. This analytical result is slightly different from that obtained by simulation Maslov (2000). However, the numerous approximations make the comparison difficult. The main point here is that some sort of mean-field approximation is natural if we assume the existence of a stationary state of the order book, and thus may help handle order book models.

Smith *et al.* (2003) also propose some sort of mean-field approximation for zero-intelligence models. In a similar model (but including a cancelation process), mean-field theory and dimensional analysis produce interesting results. For example, it is easy to see that the book depth (i.e. the number of orders) $N_e(p)$ at a price p far away from the best quotes is given by $N_e(p) = \lambda/\delta$, where λ is the rate of arrival of limit orders per unit of time and per unit of price, and δ the probability of an order being canceled per unit of time. Indeed, far from the best quotes, no market orders occur, so that if a steady state exists, the number of limit orders per time step λ must be balanced by the number of cancelations $\delta N_e(p)$ per unit of time, hence the result.

2.5.2. Explicit computation of probabilities conditional on the state of the order book. Cont *et al.* (2008) reported an original attempt at an analytical treatment of limit order books. In their model, the price is constrained to be on a grid $\{1, \ldots, N\}$. The state of the order book can then be described by the vector $X(t) = (X_1(t), \ldots, X_N(t))$, where $|X_i(t)|$ is the quantity offered in the order book at price i. Conventionally, $X_i(t)$, $i = 1, \ldots, N$, is positive on the ask side and negative on the bid side. As usual, limit orders arrive at level i at a constant rate λ_i, and market orders arrive at a constant rate μ. Finally, at level i, each order can be canceled at a rate θ_i. Using this setting, Cont *et al.* (2008) show that each event (limit order, market order, cancelation) transforms the vector X in a simple linear way. Therefore, it is shown that, under reasonable conditions, X is an ergodic Markov chain, and thus admits a stationary state. The original idea is then to use this formalism to compute conditional probabilities on the processes. More precisely, it is shown that, using a Laplace transform, one may explicitly compute the probability of an increase of the mid price conditionally on the current state of the order book.

This original contribution could allow explicit evaluation of strategies and open up new perspectives on high-frequency trading. However, it is based on a simple model that does not reproduce empirical observations such as volatility clustering. Complex models trying to include market interactions will not fit into these analytical frameworks. We review some of these models in the next section.

2.6. *Towards non-trivial behavior: Modeling market interactions*

In all the models we have reviewed thus far, flows of orders are treated as independent processes. Under certain (strong) modeling constraints, we can see the order book as a Markov chain and look for analytical results (Cont *et al.* 2008). In any case, even if the process is empirically detailed and not trivial (Mike and Farmer 2008), we work with the assumption that orders are independent and identically distributed. This very strong (and false) hypothesis is similar to the ‘representative agent’ hypothesis in Economics: orders being successively and independently submitted, we may not expect anything but regular behavior. Following the work of economists such as Kirman (1992, 1993, 2002), one has to translate the heterogeneous property of the markets into agent-based models. Agents are not identical, and not independent.

In this section we present toy models implementing mechanisms that aim at bringing heterogeneity: herding behavior on markets (Cont and Bouchaud 2000), trend-following behavior (Lux and Marchesi 2000, Preis *et al.* 2007), and threshold behavior (Cont 2007). Most of the models reviewed in this section are not order book models, since a persistent order book is not kept during the simulations. They are rather price models, where the price changes are determined by the aggregation of excess supply and demand. However, they identify essential mechanisms that may clearly explain some empirical data. Incorporating these mechanisms in an order book model has not yet been achieved, but is certainly a future possibility.

2.6.1. Herding behavior. The model presented by Cont and Bouchaud (2000) considers a market with N agents trading a given stock with price $p(t)$. At each time step, agents choose to buy or sell one unit of stock, i.e. their demand is $\phi_i(t) = \pm 1$, $i = 1, \ldots, N$, with probability a, or they are idle with probability $1 - 2a$. The price change is assumed to be linked linearly to the excess demand $D(t) = \sum_{i=1}^{N} \phi_i(t)$ with factor λ measuring the liquidity of the market:

$$p(t+1) = p(t) + \frac{1}{\lambda} \sum_{i=1}^{N} \phi_i(t). \tag{7}$$

λ can also be interpreted as the market depth, i.e. the excess demand needed to move the price by one unit. In order to evaluate the distribution of stock returns from equation (7), we need to know the joint distribution of the individual demands $(\phi_i(t))_{1 \le i \le N}$. As pointed out by the authors, if the distribution of the demand ϕ_i is independent and identically distributed with finite variance, then the Central Limit Theorem stands and the distribution of the price variation $\Delta p(t) = p(t+1) - p(t)$ will converge to a Gaussian distribution as N goes to infinity.

The idea here is to model the diffusion of the information among traders by randomly linking their demand through clusters. At each time step, agents i and j can be linked with probability $p_{ij} = p = c/N$, c being a parameter measuring the degree of clustering among agents. Therefore, an agent is linked to an average number of $(N-1)p$ other traders. Once clusters have been determined, the demands are forced to be identical among all members of a given cluster. Denoting by $n_c(t)$ the number of clusters at a given time step t, W_k the size of the kth cluster, $k = 1, \ldots, n_c(t)$, and $\phi_k = \pm 1$ its investment decision, the price variation can then be straightforwardly written as

$$\Delta p(t) = \frac{1}{\lambda} \sum_{k=1}^{n_c(t)} W_k \phi_k. \tag{8}$$

This modeling is a direct application to the field of finance of the random graph framework studied by Erdos and Renyi (1960). Kirman (1983) previously suggested it in economics. Using these previous theoretical works, and assuming that the size of a cluster W_k and the decision taken by its members $\phi_k(t)$ are independent, the authors are able to show that the distribution of the price variation at time t is the sum of $n_c(t)$ independent identically distributed random variables with heavy-tailed distributions:

$$\Delta p(t) = \frac{1}{\lambda} \sum_{k=1}^{n_c(t)} X_k, \tag{9}$$

where the density $f(x)$ of $X_k = W_k \phi_k$ is decaying as

$$f(x) \sim_{|x|\to\infty} \frac{A}{|x|^{5/2}} \mathrm{e}^{-(c-1)|x|/W_0}. \tag{10}$$

Thus, this simple toy model exhibits fat tails in the distribution of price variations, with a decay reasonably close to empirical data. Therefore, Cont and Bouchaud (2000) show that taking into account a naive mechanism of communication between agents (herding behavior) is able to drive the model out of Gaussian convergence and produce non-trivial shapes of distributions of price returns.

2.6.2. Fundamentalists and trend followers. Lux and Marchesi (2000) proposed a model very much in line with agent-based models in behavioral finance, but where the trading rules are kept sufficiently simple so that they can be identified with the presumably realistic behavior of agents. This model considers a market with N agents who can be part of two distinct groups of traders: n_f traders are 'fundamentalists', who share an exogenous idea p_f of the value of the current price p, and n_c traders are 'chartists' (or trend followers), who make assumptions concerning the price evolution based on the observed trend (mobile average). The total number of agents is constant, so that $n_f + n_c = N$ at any time. At each time step, the price can be moved up or down with a fixed jump size of ± 0.01 (a tick). The probability of going up or down is directly linked to the excess demand ED through coefficient β. The demand of each group of agents is determined as follows.

- Each fundamentalist trades a volume V_f proportional (with coefficient γ) to the deviation of the current price p from the perceived fundamental value p_f: $V_f = \gamma(p_f - p)$.
- Each chartist trades a constant volume V_c. Denoting by n_+ the number of optimistic (buyer) chartists and n_- the number of pessimistic (seller) chartists, the excess demand by the whole group of chartists is written as $(n_+ - n_-)V_c$.

Therefore, assuming that there exist noise traders on the market with random demand μ, the global excess demand can be written as

$$ED = (n_+ - n_-)V_c + n_f \gamma(p_f - p) + \mu. \tag{11}$$

The probability that the price goes up (respectively down) is then defined to be the positive (respectively negative) part of βED.

As observed by Wyart and Bouchaud (2007), fundamentalists are expected to stabilize the market, while chartists should destabilize it. In addition, following Cont and Bouchaud (2000), the authors expect the non-trivial features of the price series to results from herding behavior and transitions between groups of traders. Also referring to Kirman's work, mimicking behavior among chartists is thus proposed. The n_c chartists can change their view of the market (optimistic, pessimistic), their decision being based on a clustering process modeled by an opinion index $x = (n_+ - n_-)/n_c$ representing the weight of the majority. The probabilities π_+ and π_- of switching from one group to the other are formally written as

$$\pi_\pm = v \frac{n_c}{N} \mathrm{e}^{\pm U}, \quad U = \alpha_1 x + \alpha_2 p/v, \tag{12}$$

where v is a constant, and α_1 and α_2 reflect, respectively, the weight of the majority's opinion and the weight of the observed price in the chartists' decision. Transitions between fundamentalists and chartists are also allowed, decided by a comparison of the expected returns (see Lux and Marchesi (2000) for details).

The authors show that the distribution of returns generated by their model has excess kurtosis. Using a Hill estimator, they fit a power law to the fat tails of the distribution and observe exponents grossly ranging from 1.9 to 4.6. They also check for evidence of volatility clustering: absolute returns and squared returns exhibit a slow decay of the autocorrelation, while raw returns do not. It thus appears that such a model can grossly fit some 'stylized facts'. However, the number of parameters involved, as well as the complicated transition rules between agents, make clear identification of the sources of the phenomena and calibration to market data difficult and intractable.

Alfi *et al.* (2009a, b) provide a somewhat simplifying view of the Lux–Marchesi model. They clearly identify the fundamentalist behavior, the chartist behavior, the herding effect and the observation of the price by the agents as four essential effects of an agent-based financial model. They show that the number of agents plays a crucial role in a Lux–Marchesi-type model: more precisely, the stylized facts are reproduced only with a finite number of agents, not when the number of agents increase asymptotically, in which case the model remains in a fundamentalist regime. There is a finite-size effect that may prove important for further studies.

The role of the trend-following mechanism in producing non-trivial features in price time series was also studied by Preis *et al.* (2007). The starting point is an order book model similar to Challet and Stinchcombe (2001) and Smith *et al.* (2003): at each time step, liquidity providers submit limit orders at rate λ and liquidity takers submit market orders at rate μ. As expected, this zero-intelligence framework does not produce fat tails in the distribution of (log-)returns, nor an over-diffusive Hurst exponent. Then, a stochastic link between order placement and market trend is added: it is assumed that liquidity providers observing a trend in the market will consequently act and submit limit orders at a wider depth in the order book. Although the assumption behind such a mechanism may not be confirmed empirically (questionable symmetry in order placement is assumed) and

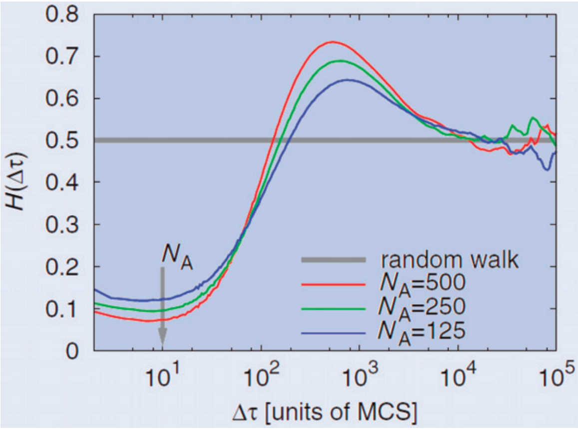

Figure 7. Hurst exponent found in the Preis model for different numbers of agents when including random demand perturbation and dynamic limit order placement depth. Reproduced from Preis *et al.* (2007).

should be discussed further, it is sufficiently interesting that it directly provides fat tails in the log-return distributions and an over-diffusive Hurst exponent $H \approx 0.6$–0.7 for medium time-scales, as shown in figure 7.

2.6.3. Threshold behavior. We finally review a model focusing primarily on reproducing the stylized fact of volatility clustering, while most of the previous models we have reviewed were mostly focused on fat tails of log returns. Cont (2007) proposes a model with a rather simple mechanism to create volatility clustering. The idea is that volatility clustering characterizes several regimes of volatility (quite periods versus bursts of activity). Instead of implementing an exogenous change of regime, the author defines the following trading rules. At each period, an agent $i \in \{1, \ldots, N\}$ can issue a buy or a sell order: $\phi_i(t) = \pm 1$. Information is represented by a series of i.i.d. Gaussian random variables (ϵ_t). This public information ϵ_t is a forecast for the value r_{t+1} of the return of the stock. Each agent $i \in \{1, \ldots, N\}$ decides whether or not to act on this information according to a threshold $\theta_i > 0$ representing its sensitivity to public information:

$$\phi_i(t) = \begin{cases} 1, & \text{if } \epsilon_i(t) > \theta_i(t), \\ 0, & \text{if } |\epsilon_i(t)| < \theta_i(t), \\ -1, & \text{if } \epsilon_i(t) < -\theta_i(t). \end{cases} \tag{13}$$

Then, once every choice is made, the price evolves according to the excess demand $D(t) = \sum_{i=1}^{N} \phi_i(t)$ in a way similar to Cont and Bouchaud (2000). At the end of each time step t, thresholds are asynchronously updated. Each agent has a probability s of updating their threshold $\theta_i(t)$. In such a case, the new threshold $\theta_i(t+1)$ is defined to be the absolute value $|r_t|$ of the return just observed. In brief,

$$\theta_i(t+1) = 1_{\{u_i(t)<s\}} |r_t| + 1_{\{u_i(t)>s\}} \theta_i(t). \tag{14}$$

The author shows that the time series simulated with such a model exhibits some realistic facts with respect to volatility. In particular, long-range correlations of absolute returns are observed. The strength of this model is that it directly links the state of the market to the decision of the trader. Such a feedback mechanism is essential in order to obtain non-trivial characteristics. Of course, the model presented by Cont (2007) is too simple to be fully calibrated on empirical data, but its mechanism could be used in a more elaborate agent-based model in order to reproduce the empirical evidence of volatility clustering.

2.7. *Remarks*

Let us attempt to make some concluding remarks concerning these developments of agent-based models for order books. In table 3 we summarize some key features of some of the order book models reviewed in this section. Among the important elements for future modeling, we mention the cancelation of orders, which is the least realistic mechanism implemented in existing models, the order book stability, which is always exogenously enforced (see our review of Mike and Farmer (2008) above), and the dependence between order flows (see, e.g., Muni Toke (2010) and reference therein). Empirical estimation of these mechanisms is still challenging.

Emphasis has been placed in this section on order book modeling, a field that is at the crossroads of many larger disciplines (market microstructure, behavioral finance and physics). Market microstructure is essential since it defines in many ways the goal of the modeling. We have pointed out that it is not a coincidence that the work of Garman (1976) was published when computerization of exchanges was about to make the electronic order book the key of all trading. The regulatory issues that motivated early studies are still very important today. Realistic order book models could be an invaluable tool in testing and evaluating the effects of regulations such as the 2005 Regulation NMS† in the USA, or the 2007 MiFID‡ in Europe.

3. Agent-based modeling for wealth distributions: Kinetic theory models

The distribution of money, wealth or income, i.e. how such quantities are shared among the population of a given country and among different countries, is a topic that has been studied by economists for a long time. The relevance of the topic here is twofold: from the point of view of the science of complex systems, wealth distributions represent a unique example of a quantitative outcome of collective behavior that can be directly compared with the predictions of theoretical models and numerical experiments. Also, there is a basic interest in wealth distributions from the social point of view, in

†National Market System.
‡Markets in Financial Instruments Directive.

Table 3. Summary of the characteristics of the reviewed limit order book models.

Model	Stigler (1964)	Garman (1976)	Bak *et al.* (1997)	Maslov (2000)	Challet and Stinchcombe (2001)	Mike and Farmer (2008)
Price range	Finite grid	Finite grid	Finite grid	Unconstrained	Unconstrained	Unconstrained
Clock	Trade time	Physical time	Aggregated time	Event time	Aggregated time	Aggregated time
Flows/agents	One zero-intelligence agent/one flow	One zero-intelligence agent/two flows (buy/sell)	N agents each owning one limit order	One zero-intelligence flow (limit order with fixed probability, else market order)	One zero-intelligence agent/one flow	One zero-intelligence agent/one flow
Limit orders	Uniform distribution on the price grid	Two Poisson processes for buy and sell orders	Moving at each time step by one tick	Uniformly distributed in a finite interval around the last price	Normally distributed around best quote	Student-distributed around best quote
Market orders	Defined as crossing limit orders	Defined as crossing limit orders	Defined as crossing limit orders	Submitted as such	Defined as crossing limit orders	Defined as crossing limit orders
Cancelation orders	Pending orders are canceled after a fixed number of time steps	None	None (constant number of pending orders)	Pending orders are canceled after a fixed number of time steps	Pending orders can be canceled with fixed probability at each time step	Pending orders can be canceled with three-parameter conditional probability at each time step
Volume	Unit	Unit	Unit	Unit	Unit	Unit
Order signs	Independent	Independent	Independent	Independent	Independent	Correlated with a fractional Brownian motion
Claimed results	Return distribution is power law 0.3/cut-off because finite grid	Microstructure is responsible for negative correlation of consecutive price changes	No fat tails for returns/Hurst exponent 1/4 for price increments	Fat tails for distributions of returns/Hurst exponent 1/4	Hurst exponent 1/4 for short time scales, tending to 1/2 for longer time scales	Fat tail distributions of returns/realistic spread distribution/unstable order book

Table 4. Gini indices (in percent) of some countries (from *Human Development Indicators of the United Nations Human Development Report*, 2004, pp. 50–53. Available at http://hdr.undp.org/en/reports/global/hdr2004. More recent data are also available from their website).

Denmark	24.7
Japan	24.9
Sweden	25.0
Norway	25.8
Germany	28.3
India	32.5
France	32.7
Australia	35.2
UK	36.0
USA	40.8
Hong Kong	43.4
China	44.7
Russia	45.6
Mexico	54.6
Chile	57.1
Brazil	59.1
South Africa	59.3
Botswana	63.0
Namibia	70.7

particular their degree of (in)equality. To this aim, the Gini coefficient (or the Gini index, if expressed as a percentage), developed by the Italian statistician Corrado Gini, represents a concept commonly employed to measure inequality of wealth distributions or, in general, how uneven a given distribution is. For a cumulative distribution function $F(y)$ that is piecewise differentiable, has a finite mean μ, and is zero for $y<0$, the Gini coefficient is defined as

$$G = 1 - \frac{1}{\mu}\int_0^\infty \mathrm{d}y\,(1-F(y))^2 = \frac{1}{\mu}\int_0^\infty \mathrm{d}y\,F(y)(1-F(y)). \tag{15}$$

It can also be interpreted statistically as half the relative mean difference. Thus the Gini coefficient is a number between 0 and 1, where 0 corresponds to perfect equality (where everyone has the same income) and 1 corresponds to perfect inequality (where one person has all the income, and everyone else has zero income). Some values of G for some countries are listed in table 4.

Let us start by considering the basic economic quantities: money, wealth and income.

3.1. Money, wealth and income

A common definition of *money* suggests that it is the "[c]ommodity accepted by general consent as medium of economics exchange".† In fact, money circulates from one economic agent (which can represent an individual, firm, country, etc.) to another, thus facilitating trade. It is "something which all other goods or services are traded for" (for details, see Shostak (2000)). Throughout history, various commodities have been used as money, termed 'commodity money', which include, for example, rare seashells or beads and cattle (such as cows in India). Recently, 'commodity money' has been replaced by other forms referred to as 'fiat money', which have gradually become the most common, such as metal coins and paper notes. Nowadays, other forms of money, such as electronic money, have become the most frequent form used to carry out transactions. In any case, the most relevant points concerning the money employed are its basic functions, which, according to standard economic theory, are:

- to serve as a medium of exchange that is universally accepted in trade for goods and services;
- to act as a measure of value, making possible the determination of prices and the calculation of costs, or profit and loss;
- to serve as a standard of deferred payments, i.e. a tool for the payment of debt or the unit in which loans are made and future transactions are fixed; and
- to serve as a means of storing wealth not immediately required for use.

A related feature relevant to the present investigation is that money is the medium in which prices or the values of all commodities as well as costs, profits, and transactions can be determined or expressed. *Wealth* is usually understood as things that have economic utility (monetary value or value of exchange), or material goods or property; it also represents the abundance of objects of value (or riches) and the state of having accumulated these objects. For our purposes, it is important to bear in mind that wealth can be measured in terms of money. Also *income*, defined by Case and Fair (2008) as "the sum of all the wages, salaries, profits, interests payments, rents and other forms of earnings received … in a given period of time", is a quantity that can be measured in terms of money (per unit time).

3.2. Modeling wealth distributions

It was first observed by Pareto (1897) that, in an economy, the higher end of the distribution of income $f(x)$ follows a power-law,

$$f(x) \sim x^{-1-\alpha}, \tag{16}$$

with α, now known as the Pareto exponent, estimated by him to be $\alpha \approx 3/2$. For the last 100 years the value of $\alpha \sim 3/2$ seems to have changed little in time and across the various capitalist economies (see Yakovenko and Rosser (2009) and references therein).

Gibrat (1931) clarified that Pareto's law is valid only for the high-income range, whereas for the middle-income range he suggested that the income distribution is

†*Encyclopædia Britannica*. Retrieved 17 June 2010 from Encyclopædia Britannica Online.

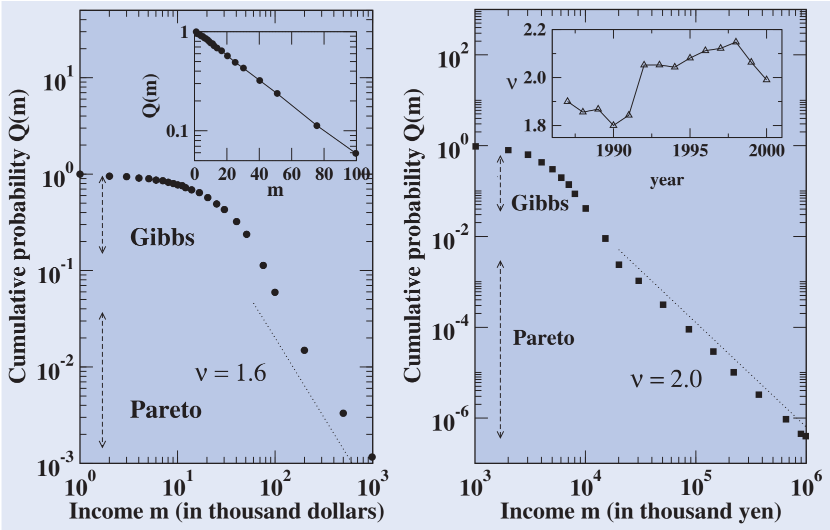

Figure 8. Income distributions in the US (left) and Japan (right). Reproduced and adapted from Chakrabarti and Chatterjee (2003). Available at arXiv:cond-mat/0302147.

described by a log-normal probability density

$$f(x) \sim \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{\log^2(x/x_0)}{2\sigma^2}\right\}, \tag{17}$$

where $\log(x_0) = \langle \log(x) \rangle$ is the mean value of the logarithmic variable and $\sigma^2 = \langle [\log(x) - \log(x_0)]^2 \rangle$ the corresponding variance. The factor $\beta = 1/\sqrt{2\sigma^2}$, also know an the Gibrat index, measures the equality of the distribution.

More recent empirical studies on income distribution have been carried out by physicists, e.g. those by Dragulescu and Yakovenko (2001a, b) for the UK and US, by Fujiwara *et al.* (2003) for Japan, and by Nirei and Souma (2007) for the US and Japan. For an overview, see Yakovenko and Rosser (2009). The distributions obtained have been shown to follow either the log-normal (Gamma-like) or power-law types, depending on the range of wealth, as shown in figure 8.

One of the current challenges is to write down the 'microscopic equation' that governs the dynamics of the evolution of wealth distributions, possibly predicting the observed shape of wealth distributions, including the exponential law at intermediate values of wealth as well as the century-old Pareto law. To this aim, several studies have been performed to investigate the characteristics of the real income distribution and provide theoretical models or explanations (see, e.g., reviews by Lux (2005), Chatterjee and Chakrabarti (2007) and Yakovenko and Rosser (2009)).

The model of Gibrat (1931) and other models formulated in terms of a Langevin equation for a single wealth variable, subject to multiplicative noise (Mandelbrot 1960, Levy and Solomon 1996, Sornette 1998, Burda *et al.* 2003), can lead to equilibrium wealth distributions with a power-law tail, since they converge towards a log-normal distribution. However, the fit of real wealth distributions does not turn out to be as good as that obtained using, for example, a Γ or β distribution, in particular due to the too large asymptotic variances (Angle 1986). Other models use a different approach and describe the wealth dynamics as a wealth flow due to exchanges between (pairs of) basic units. In this respect, such models are basically different from the class of models formulated in terms of a Langevin equation for a single wealth variable. For example, Solomon and Levy (1996) studied the generalized Lotka–Volterra equations in relation to a power-law wealth distribution. Ispolatov *et al.* (1998) studied random exchange models of wealth distributions. Other models describing wealth exchange have been formulated using matrix theory (Gupta 2006), the master equation (Bouchaud and Mezard 2000, Dragulescu and Yakovenko 2000, Ferrero 2004), the Boltzmann equation approach (Dragulescu and Yakovenko 2000, Slanina 2004, Cordier *et al.* 2005, Repetowicz *et al.* 2005, Düring and Toscani 2007, Matthes and Toscani 2007, Düring *et al.* 2008), or Markov chains (Scalas *et al.* 2006, 2007, Garibaldi *et al.* 2007). It should be mentioned that one of the earliest modeling efforts was that of Champernowne (1953). Since then, many economists (Gabaix (1999) and Benhabib and Bisin (2009), among others) have also studied mechanisms for power laws, and distributions of wealth.

In the two following sections we consider in greater detail a class of models usually referred to as *kinetic wealth exchange models* (KWEM), formulated through finite-time difference stochastic equations (Angle 1986, 2002, 2006, Chakraborti and Chakrabarti 2000, Dragulescu and Yakovenk 2000, Chakraborti 2002, Hayes 2002, Chatterjee *et al.* 2003, Das and Yarlagadda 2003, Iglesias *et al.* 2003, 2004, Scafetta *et al.* 2004, Ausloos and Pekalski 2007). From the studies carried out using wealth-exchange models, it emerges that it is possible to use them to generate power-law distributions.

3.3. Homogeneous kinetic wealth exchange models

Here and in the following section we consider KWEMs, which are statistical models for a closed economy. Their goal, rather then describing the market dynamics in terms of intelligent agents, is to predict the time evolution of the distribution of some main quantity, such as wealth, by studying the corresponding flow process among individuals. The underlying idea is that however complicated the detailed rules of wealth exchanges are, their average behavior can be described in a relatively simple way and will share some universal properties with other transport processes, due to the general conservation constraints and the effect of the fluctuations due to the environment or associated with individual behavior. Here, there is a clear analogy with the general theory of transport phenomena (e.g. of energy).

In these models the states of agents are defined in terms of the wealth variables $\{x_n\}$, $n=1,2,\ldots,N$. Evolution of the system is carried out according to a trading rule between agents, which, to obtain the final equilibrium distribution, can be interpreted as the actual time evolution of the agent states as well as a Monte Carlo optimization. The algorithm is based on a simple update rule performed at each time step t, when two agents i and j are extracted randomly and an amount of wealth Δx is exchanged,

$$\begin{aligned} x_i' &= x_i - \Delta x, \\ x_j' &= x_j + \Delta x. \end{aligned} \tag{18}$$

Note that the quantity x is conserved during single transactions, $x_i' + x_j' = x_i + x_j$, where $x_i = x_i(t)$ and $x_j = x_j(t)$ are the agent wealth values before the transaction, whereas $x_i' = x_i(t+1)$ and $x_j' = x_j(t+1)$ are the final wealth values after the transaction. Several rules have been studied for the model defined by equation (18). It is noteworthy that although this theory was originally derived from the entropy maximization principle of statistical mechanics, it has recently been shown that the same rule could also be derived from the utility maximization principle, following a standard exchange model with Cobb–Douglas utility function (as explained below), which brings physics and economics together.

3.3.1. Exchange models without saving. In a simple version of KWEM considered in the studies of Bennati (1988a, b, 1993) and also studied by Dragulescu and Yakovenko (2000), the money difference Δx in equation (18) is assumed to have a constant value, $\Delta x = \Delta x_0$. Together with the constraint that transactions can take place only if $x_i' > 0$ and $x_j' > 0$, this leads to an equilibrium exponential distribution (see the curve for $\lambda = 0$ in figure 9).

Various other trading rules were studied by Dragulescu and Yakovenko (2000), choosing Δx as a random fraction of the average money between the two agents, $\Delta x = \epsilon(x_i + x_j)/2$, corresponding to $\Delta x = (1-\epsilon)x_i - \epsilon x_j$ in equation (18), or of the average money of the whole system, $\Delta x = \epsilon \langle x \rangle$.

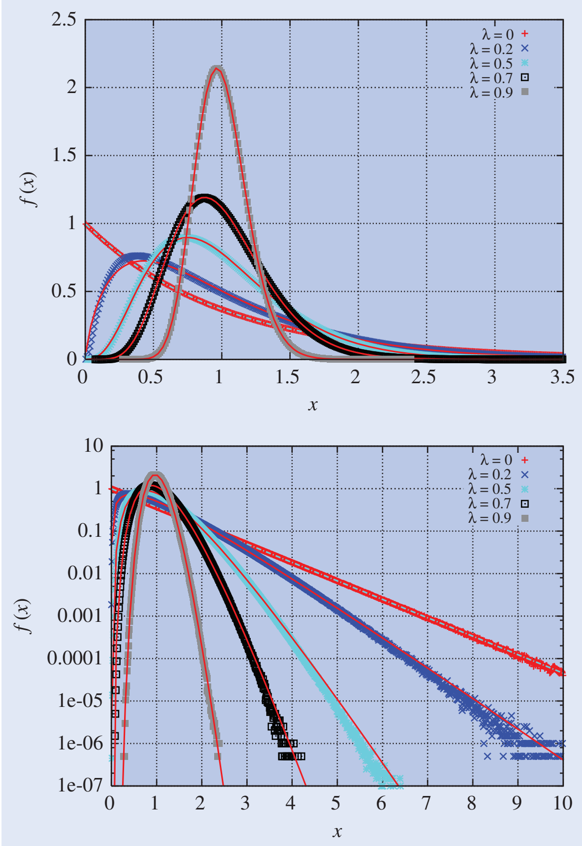

Figure 9. Probability density for wealth x. The curve for $\lambda=0$ is the Boltzmann function $f(x) = \langle x \rangle^{-1} \exp(-x/\langle x \rangle)$ for the basic model of section 3.3.1. The other curves correspond to the global saving propensity $\lambda > 0$ (see section 3.3.2.)

The models mentioned, as well as more complicated models (Dragulescu and Yakovenko 2000), lead to an equilibrium wealth distribution with an exponential tail

$$f(x) \sim \beta \exp(-\beta x), \tag{19}$$

with the effective temperature $1/\beta$ of the order of the average wealth, $\beta^{-1} = \langle x \rangle$. This result is largely independent of the details of the model, e.g. the multi-agent nature of the interaction, the initial conditions, and the random or consecutive order of extraction of the interacting agents. The Boltzmann distribution is characterized by a majority of poor agents and a few rich agents (due to the exponential tail), and has a Gini coefficient of 0.5.

3.3.2. Exchange models with saving. As a generalization and more realistic version of the basic exchange models, a saving criterion can be introduced. Angle (1983), motivated by the surplus theory, introduced a unidirectional model of wealth exchange, in which only a fraction of wealth smaller than one can pass from one agent to the other, with $\Delta x = \epsilon x_i$ or $(-\omega x_j)$, where the direction of the flow is determined by the agent wealth (Angle 1983, 1986). Later, Angle introduced the One-Parameter Inequality Process (OPIP) where a constant fraction

$1-\omega$ is saved before the transaction (Angle 2002) by the agent whose wealth decreases, defined by an exchanged wealth amount $\Delta x = \omega x_i$ or $-\omega x_j$, again with the direction of the transaction determined by the relative difference between the agents' wealth.

A 'saving parameter' $0 < \lambda < 1$, representing the fraction of wealth *saved*, was introduced in the model of Chakraborti and Chakrabarti (2000). In this model (CC), wealth flows simultaneously to and from each agent during a single transaction, the dynamics being defined by the equations

$$\begin{aligned} x_i' &= \lambda x_i + \epsilon(1-\lambda)(x_i + x_j), \\ x_j' &= \lambda x_j + (1-\epsilon)(1-\lambda)(x_i + x_j), \end{aligned} \tag{20}$$

or, equivalently, by Δx in (18), given by

$$\Delta x = (1-\lambda)[(1-\epsilon)x_i - \epsilon x_j]. \tag{21}$$

These models, apart from the OPIP model of Angle which has the remarkable property of leading to a power law in a suitable range of ω, can be well-fitted by a Γ distribution. The Γ distribution is characterized by a mode $x_m > 0$, in agreement with real data of wealth and income distributions (Dragulescu and Yakovenko 2001a, Sala-i Martin 2002, Sala-i Martin and Mohapatra 2002, Aoyama *et al.* 2003, Ferrero 2004, Silva and Yakovenko 2005). Furthermore, the limit for small x is zero, i.e. $P(x \to 0) \to 0$ (see the example in figure 9). In the particular case of the model of Chakraborti and Chakrabarti (2000), the explicit distribution is well-fitted by

$$\begin{aligned} f(x) &= n\langle x\rangle^{-1}\gamma_n(nx/\langle x\rangle) \\ &= \frac{1}{\Gamma(n)}\frac{n}{\langle x\rangle}\left(\frac{nx}{\langle x\rangle}\right)^{n-1}\exp\left(-\frac{nx}{\langle x\rangle}\right), \end{aligned} \tag{22}$$

$$n(\lambda) \equiv \frac{D_\lambda}{2} = 1 + \frac{3\lambda}{1-\lambda}, \tag{23}$$

where $\gamma_n(\xi)$ is the standard Γ distribution. This particular functional form was conjectured on the basis of the excellent fitting provided to numerical data (Angle 1983, 1986, Patriarca *et al.* 2004a, b, Heinsalu *et al.* 2009). For more information and a comparison of similar fittings for different models, see Patriarca *et al.* (2010). Very recently, Lallouache *et al.* (2010) showed, using the distributional form of the equation and moment calculations, that, strictly speaking, the Gamma distribution is not the solution of equation (20), confirming the earlier results of Repetowicz *et al.* (2005). However, the Gamma distribution is a very very good approximation.

The ubiquitous presence of Γ functions in the solutions of kinetic models (see also heterogeneous models below) suggests a close analogy with the kinetic theory of gases. In fact, interpreting $D_\lambda = 2n$ as an effective dimension, the variable x as kinetic energy, and introducing the effective temperature $\beta^{-1} \equiv T_\lambda = \langle x\rangle/2D_\lambda$ according to the equipartition theorem, equations (22) and (23) define the canonical distribution $\beta\gamma_n(\beta x)$ for the kinetic energy of a gas in $D_\lambda = 2n$ dimensions (see Patriarca *et al.* (2004a) for details). The analogy is illustrated in table 5 and the dependencies of $D_\lambda = 2n$ and of $\beta^{-1} = T_\lambda$ on the saving parameter λ are shown in figure 10.

Table 5. Analogy between the kinetic theory of gases and the kinetic exchange model of wealth.

	Kinetic model	Economy model
Variable	K (kinetic energy)	x (wealth)
Units	N particles	N agents
Interaction	Collisions	Trades
Dimension	Integer D	Real number D_λ
Temperature definition	$k_B T = 2\langle K\rangle/D$	$T_\lambda = 2\langle x\rangle/D_\lambda$
Reduced variable	$\xi = K/k_B T$	$\xi = x/T_\lambda$
Equilibrium distribution	$f(\xi) = \gamma_{D/2}(\xi)$	$f(\xi) = \gamma_{D_\lambda/2}(\xi)$

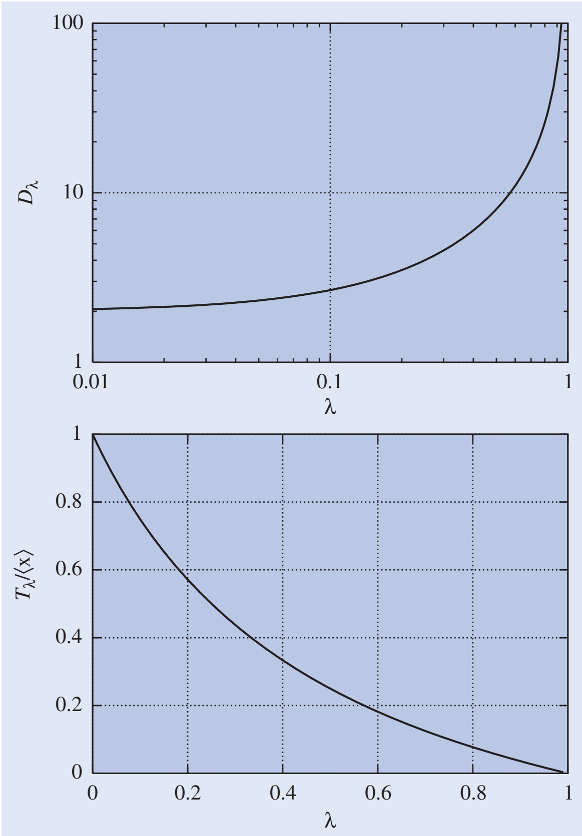

Figure 10. Effective dimension D_λ and temperature T as a function of the saving parameter λ.

The exponential distribution is recovered as a special case, for $n = 1$. In the limit $\lambda \to 1$, i.e. for $n \to \infty$, the distribution $f(x)$ tends to a Dirac δ function, as shown by Patriarca *et al.* (2004a) and illustrated qualitatively in figure 9. This shows that a large saving criterion leads to a final state in which economic agents tend to have similar amounts of money and, in the limit $\lambda \to 1$, exactly the same amount $\langle x\rangle$.

The equivalence between a kinetic wealth-exchange model with saving propensity $\lambda \geq 0$ and an N-particle system in a space with dimension $D_\lambda \geq 2$ is suggested by simple considerations concerning the kinetics of collision processes between two molecules. In one dimension, particles undergo head-on collisions in which all the kinetic energy can be exchanged. In a larger number of dimensions the two particles will, in general, not travel along exactly the same line, in opposite verses, and only a fraction of the energy can be exchanged. It can be shown that, during a binary elastic collision in D dimensions, only a fraction $1/D$ of the total kinetic energy is exchanged, on average, for kinematic reasons (see Chakraborti and Patriarca (2008) for details). The same $1/D$ dependence is in fact obtained by inverting equation (23), which provides for the fraction of exchanged wealth $1-\lambda=6/(D_\lambda+4)$.

Not all homogeneous models lead to distributions with an exponential tail. For instance, in the model studied by Chakraborti (2002) an agent i can lose all his wealth, thus becoming unable to trade again: after a sufficient number of transactions, only one trader survives in the market and owns the entire wealth. The equilibrium distribution has a very different shape, as explained below.

In the toy model it is assumed that both economic agents i and j invest the same amount $x_{\min}$, which is taken as the minimum wealth between the two agents, $x_{\min}=\min\{x_i, x_j\}$. The wealth after the trade is $x_i'=x_i+\Delta x$ and $x_j'=x_j-\Delta x$, where $\Delta x=(2\epsilon-1)x_{\min}$. We note that once an agent has lost all his wealth, he is unable to trade because x_min has become zero. Thus, a trader is effectively driven out of the market once he loses all his wealth. In this way, after a sufficient number of transactions, only one trader survives in the market with the entire amount of wealth, whereas the rest of the traders have zero wealth. In this toy model, only one agent has the entire money of the market and the rest of the traders have zero money, which corresponds to a distribution with Gini coefficient equal to unity.

A situation is said to be Pareto-optimal "if by reallocation you cannot make someone better off without making someone else worse off". In Pareto's own words:

> "We will say that the members of a collectivity enjoy maximum ophelimity in a certain position when it is impossible to find a way of moving from that position very slightly in such a manner that the ophelimity enjoyed by each of the individuals of that collectivity increases or decreases. That is to say, any small displacement in departing from that position necessarily has the effect of increasing the ophelimity which certain individuals enjoy, and decreasing that which others enjoy, of being agreeable to some, and disagreeable to others."— Vilfredo Pareto, *Manual of Political Economy*, 1906, p. 261.

However, as Sen (1971) notes, *an economy can be Pareto-optimal, yet still 'perfectly disgusting' by any ethical standards*. It is important to note that Pareto-optimality is merely a descriptive term, a property of an 'allocation', and there are no ethical propositions concerning the desirability of such allocations inherent within that notion. Thus, in other words, there is nothing inherent in Pareto-optimality that implies the maximization of social welfare.

This simple toy model thus also produces a Pareto-optimal state (it will be impossible to raise the wellbeing of anyone except the *winner*, i.e. the agent with all the money, and *vice versa*), but the situation is economically undesirable as far as social welfare is concerned! Note also that, as mentioned above, the OPIP model of Angle (2006, 2002), for example, depending on the model parameters, can also produce a power-law tail. Another general way to produce a power-law tail in the equilibrium distribution seems to be by diversifying the agents, i.e. to consider heterogeneous models, discussed below.

3.4. *Heterogeneous kinetic wealth exchange models*

3.4.1. Random saving propensities. The models considered above assume that all the agents have the same statistical properties. The corresponding equilibrium wealth distribution has, in most cases, an exponential tail, a form that well interpolates real data at small and intermediate values of wealth. However, it is possible to conceive generalized models that lead to even more realistic equilibrium wealth distributions. This is the case when agents are diversified by assigning different values of the saving parameter. For instance, Angle (2002) studied a model with a trading rule where diversified parameters $\{\omega_i\}$ occur,

$$\Delta x = \omega_i \epsilon x_i \quad \text{or} \quad -\omega_j \epsilon x_j, \tag{24}$$

with the direction of wealth flow determined by the wealth of agents i and j. Diversified saving parameters were independently introduced by Chatterjee *et al.* (2003, 2004) by generalizing the model introduced by Chakraborti and Chakrabarti (2000):

$$\begin{aligned} x_i' &= \lambda_i x_i + \epsilon[(1-\lambda_i)x_i + (1-\lambda_j)x_j], \\ x_j' &= \lambda x_j + (1-\epsilon)[(1-\lambda_i)x_i + (1-\lambda_j)x_j], \end{aligned} \tag{25}$$

corresponding to

$$\Delta x = (1-\epsilon)(1-\lambda_i)x_i - \epsilon(1-\lambda_j)x_j. \tag{26}$$

The surprising result is that if the parameters $\{\lambda_i\}$ are suitably diversified, a power law appears in the equilibrium wealth distribution (see figure 11). In particular, if the λ_i are uniformly distributed in $(0, 1)$, the wealth distribution exhibits a robust power-law tail,

$$f(x) \propto x^{-\alpha-1}, \tag{27}$$

with the Pareto exponent $\alpha=1$ largely independent of the details of the λ distribution. It should be noted that the exponent value of unity is strictly for the tail end of the distribution and not for small values of the income or wealth (where the distribution remains exponential). Also, for finite number N of agents, there is always an exponential (in N) cut-off at the tail end of the distribution. This result is supported by independent theoretical considerations based on various approaches, such as a

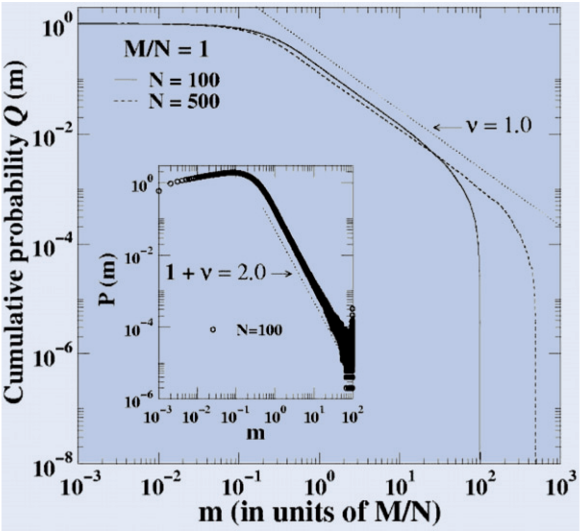

Figure 11. Results for randomly assigned saving parameters. Reproduced and adapted from Chakrabarti and Chatterjee (2003). Available at arXiv:cond-mat/0302147.

mean-field theory approach (Mohanty 2006) (see below for further details) or the Boltzmann equation (Das and Yarlagadda 2003, 2005, Chatterjee *et al.* 2005a, Repetowicz *et al.* 2005). For a derivation of the Pareto law from variational principles, using the KWEM context, see Chakraborti and Patriarca (2009).

3.4.2. Power-law distribution as an overlap of Gamma distributions. A remarkable feature of the equilibrium wealth distribution obtained from heterogeneous models, reported by Chatterjee *et al.* (2004), is that the individual wealth distribution $f_i(x)$ of the generic ith agent with saving parameter λ_i has a well-defined mode and exponential tail, in spite of the resulting power-law tail of the marginal distribution $f(x) = \sum_i f_i(x)$. In fact, Patriarca *et al.* (2005) found by numerical simulation that the marginal distribution $f(x)$ can be resolved as an overlap of individual Gamma distributions with λ-dependent parameters; furthermore, the mode and the average value of the distributions $f_i(x)$ both diverge for $\lambda \to 1$ as $\langle x(\lambda)\rangle \sim 1/(1-\lambda)$ (Chatterjee *et al.* 2004, Patriarca *et al.* 2005). This fact was justified theoretically by Mohanty (2006). Consider the evolution equations (25). In the mean-field approximation, one can consider that each agents i has an (average) wealth $\langle x_i\rangle = y_i$ and replace the random number ϵ by its average value $\langle\epsilon\rangle = 1/2$. Denoting by y_{ij} the new wealth of agent i, due to the interaction with agent j, from equations (25) one obtains

$$y_{ij} = (1/2)(1+\lambda_i)\,y_i + (1/2)(1-\lambda_j)\,y_j. \tag{28}$$

At equilibrium, for consistency, averaging over all the interactions must return y_i,

$$y_i = \sum_j y_{ij}/N. \tag{29}$$

Then summing equation (28) over j and dividing by the number of agents N, one has

$$(1-\lambda_i)\,y_i = \langle(1-\lambda)\,y\rangle, \tag{30}$$

where $\langle(1-\lambda)y\rangle = \sum_j(1-\lambda_j)y_j/N$. Since the right-hand side is independent of i and this relation holds for arbitrary distributions of λ_i, the solution is

$$y_i = \frac{C}{1-\lambda_i}, \tag{31}$$

where C is a constant. Besides proving the dependence of $y_i = \langle x_i\rangle$ on λ_i, this relation also demonstrates the existence of a power-law tail in the equilibrium distribution. If, in the continuous limit, λ is distributed in (0, 1) with density $\phi(\lambda)$ $(0 \le \lambda < 1)$, then using (31) the (average) wealth distribution is given by

$$f(y) = \phi(\lambda)\frac{\mathrm{d}\lambda}{\mathrm{d}y} = \phi(1 - C/x)\frac{C}{y^2}. \tag{32}$$

Figure 12 illustrates the phenomenon for a system of $N = 1000$ agents with random saving propensities uniformly distributed between 0 and 1. The figure confirms the importance of agents with λ close to 1 for producing a power-law probability distribution (Chatterjee *et al.* 2004, Heinsalu *et al.* 2009).

However, when considering values of λ sufficiently close to 1, the power law can break down for (at least) two reasons. The first, illustrated in figure 12 (bottom right), is that the power law can be resolved into almost disjoint contributions representing the wealth distributions of single agents. This follows from the finite number of agents used and the fact that the distance between the average values of the distributions corresponding to two consecutive values of λ increases faster than the corresponding widths (Chatterjee *et al.* 2005b, Patriarca *et al.* 2005). The second reason is due to the finite cut-off λ_M, always present in a numerical simulation. However, to study this effect, one has to consider a system with a sufficiently large number of agents that it is not possible to resolve the wealth distributions of single agents for the sub-intervals of λ considered. This was done by Patriarca *et al.* (2006) using a system with $N = 10^5$ agents with saving parameters distributed uniformly between 0 and λ_M. The results are shown in figure 13, where the curves from left to right correspond to increasing values of the cut-off λ_M from 0.9 to 0.9997.

The transition from an exponential to a power-law tail takes place continuously as the cut-off λ_M is increased beyond a critical value $\lambda_M \approx 0.9$ towards $\lambda_M = 1$, through enlargement of the x interval in which the power-law is observed.

3.4.3. Relaxation process. Relaxation in systems with constant λ has already been studied by Chakraborti and Chakrabarti (2000), where a systematic increase of the relaxation time with λ, and eventually a divergence for $\lambda \to 1$, was found. In fact, for $\lambda = 1$, no exchanges occurs and the system is frozen.

The relaxation time scale of a heterogeneous system has been studied by Patriarca *et al.* (2007). The system is observed to relax towards the same equilibrium wealth distribution from any given arbitrary initial distribution of wealth. If time is measured by the number of

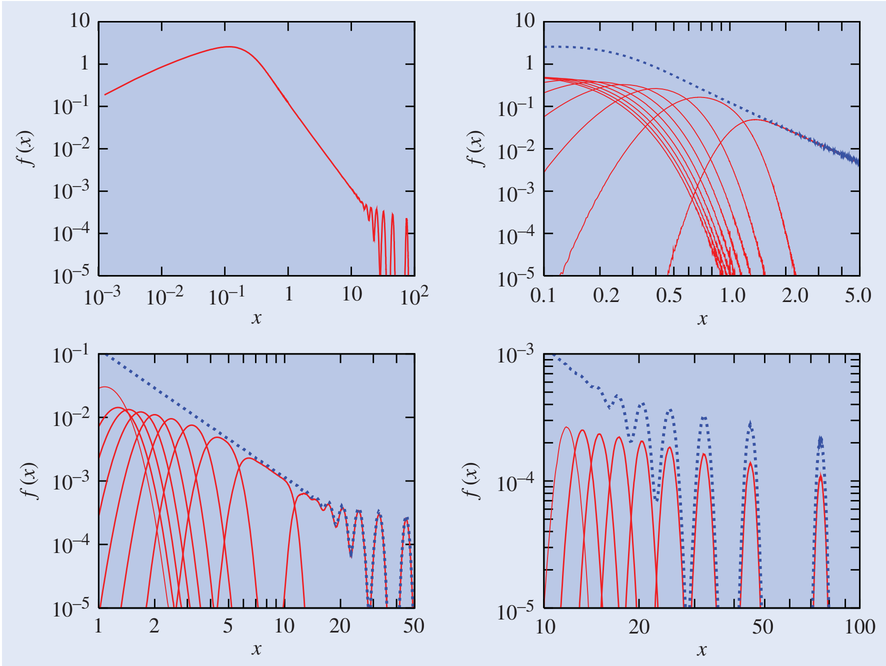

Figure 12. Wealth distribution in a system of 1000 agents with saving propensities uniformly distributed in the interval $0<\lambda<1$. Top left: marginal distribution. Top right: marginal distribution (dotted line) and distributions of wealth of agents with $\lambda \in (j\Delta\lambda, (j+1)\Delta\lambda)$, $\Delta\lambda = 0.1$, $j = 0, \ldots, 9$ (continuous lines). Bottom-left: the distribution of wealth of agents with $\lambda \in (0.9, 1)$ has been further resolved into contributions from subintervals $\lambda \in (0.9 + j\Delta\lambda, 0.9 + (j+1)\Delta\lambda)$, $\Delta\lambda = 0.01$. Bottom-right: the partial distribution of the wealth of agents with $\lambda \in (0.99, 1)$ has been further resolved into those from subintervals $\lambda \in (0.99 + j\Delta\lambda, 0.99 + (j+1)\Delta\lambda)$, $\Delta\lambda = 0.001$. Reproduced from Patriarca *et al.* (2006).

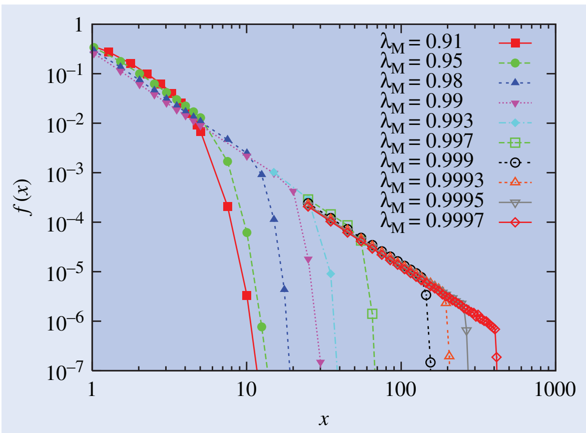

Figure 13. Wealth distribution obtained for the uniform saving propensity distributions of 10^5 agents in the interval $(0, \lambda_M)$. Reproduced from Patriarca *et al.* (2006).

transactions n_t, the time scale is proportional to the number of agents N, i.e. defining time t as the ratio $t = n_t/N$ between the number of trades and the total number of agents N (corresponding to one Monte Carlo cycle or one sweep in molecular dynamics simulations), and the dynamics and the relaxation process become independent of N. The existence of a natural time scale independent of the system size provides a foundation for using simulations of systems with finite N in order to infer properties of systems with continuous saving propensity distributions and $N \to \infty$.

In a system with uniformly distributed λ, the wealth distribution of each agent i with saving parameter λ_i relaxes towards different states with characteristic shapes $f_i(x)$ (Chatterjee *et al.* 2005b, Patriarca *et al.* 2005, 2006) with different relaxation times τ_i (Patriarca *et al.* 2007). The differences in the relaxation processes can be related to the different relative wealth exchange rates, which by direct inspection of the evolution equations appear to be proportional to $1-\lambda_i$. Thus, in general, higher saving propensities are expected to be associated with slower relaxation processes with a relaxation time $\propto 1/(1-\lambda)$.

It is also possible to obtain the relaxation time distribution. If the saving parameters are distributed in $(0, 1)$ with density $\phi(\lambda)$, it follows from probability conservation that $\tilde{f}(\bar{x})\mathrm{d}\bar{x} = \phi(\lambda)\mathrm{d}\lambda$, where $\bar{x} \equiv \langle x \rangle_\lambda$ and $\tilde{f}(\bar{x})$ is the corresponding density of the average wealth values. In the case of uniformly distributed saving propensities, one obtains

$$\tilde{f}(\bar{x}) = \phi(\lambda)\frac{\mathrm{d}\lambda(\bar{x})}{\mathrm{d}\bar{x}} = \phi\left(1-\frac{k}{\bar{x}}\right)\frac{k}{\bar{x}^2}, \tag{33}$$

showing that a uniform saving propensity distribution leads to a power law $\tilde{f}(\bar{x}) \sim 1/\bar{x}^2$ in the (average) wealth distribution. In a similar way it is possible to obtain the associated distribution of relaxation times $\psi(\tau)$ for the global relaxation process from the relation $\tau_i \propto 1/(1-\lambda_i)$,

$$\psi(\tau) = \phi(\lambda)\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau} \propto \phi\left(1-\frac{\tau'}{\tau}\right)\frac{\tau'}{\tau^2}, \tag{34}$$

where τ' is a proportionality factor. Therefore, $\psi(\tau)$ and $\tilde{f}(\bar{x})$ are characterized by power-law tails in τ and $\bar{x}$, respectively, *with the same Pareto exponent*.

In conclusion, the role of the λ cut-off is also related to the relaxation process. This means that the slowest convergence rate is determined by the cut-off and is $\propto 1-\lambda_M$. In numerical simulations of heterogeneous KWEMs, as well as in real wealth distributions, the cut-off is necessarily finite, so that the convergence is fast (Gupta 2008). On the other hand, if considering a hypothetical wealth distribution with a power law extending to infinite values of x, one cannot find a fast relaxation, due to the infinite time scale of the system, owing to the agents with $\lambda = 1$.

3.5. *Microeconomic formulation of kinetic theory models*

Very recently, Chakrabarti and Chakrabarti (2009) studied the framework based on microeconomic theory from which kinetic theory market models can be addressed. They derived the moments of the model of Chakraborti and Chakrabarti (2000) and reproduced the exchange equations used in the model (with fixed savings parameter). In the framework considered, the utility function deals with the behavior of the agents in an exchange economy. They start by considering two exchange economies, where each agent produces a single perishable commodity. Each of these goods is different and money exists in the economy to simply facilitate transactions. Each of these agents is endowed with an initial amount of money $M_1 = m_1(t)$ and $M_2 = m_2(t)$. Let agent 1 produce Q_1 amount of commodity 1 only, and agent 2 produce Q_2 amount of commodity 2 only. At each time step t, the two agents meet randomly to carry out transactions according to their utility maximization principle.

The utility functions are defined as follows. For agent 1, $U_1(x_1, x_2, m_1) = x_1^{\alpha_1} x_2^{\alpha_2} m_1^{\alpha_m}$, and for agent 2, $U_2(y_1, y_2, m_2) = y_1^{\alpha_1} y_2^{\alpha_2} m_2^{\alpha_m}$, where the arguments in both of the utility functions are consumption of the first (i.e. x_1 and y_1) and second good (i.e. x_2 and y_2) and the amount of money in their possession, respectively. For simplicity, they assume that the utility functions are of the above Cobb–Douglas form with the sum of the powers normalized to 1, i.e. $\alpha_1 + \alpha_2 + \alpha_m = 1$.

Let the commodity prices to be determined in the market be denoted by p_1 and p_2. The budget constraints are as follows. For agent 1 the budget constraint is $p_1 x_1 + p_2 x_2 + m_1 \le M_1 + p_1 Q_1$, and similarly for agent 2 the constraint is $p_1 y_1 + p_2 y_2 + m_2 \le M_2 + p_2 Q_2$, which means that the amount that agent 1 can spend on consuming x_1 and x_2 added to the amount of money that he holds after trading at time $t+1$ (i.e. m_1) cannot exceed the amount of money that he has at time t (i.e. M_1) added to what he earns by selling the good he produces (i.e. Q_1), and the same is true for agent 2.

The basic idea is that both of the agents try to maximize their respective utility subject to their respective budget constraints and the *invisible hand* of the market, that is the price mechanism works to clear the market for both goods (i.e. total demand equals total supply for both goods at the equilibrium prices), which means that agent 1's problem is to maximize his utility subject to his budget constraint, i.e. maximize $U_1(x_1, x_2, m_1)$ subject to $p_1 \cdot x_1 + p_2 \cdot x_2 + m_1 = M_1 + p_1 \cdot Q_1$. Similarly, for agent 2 the problem is to maximize $U_1(y_1, y_2, m_2)$ subject to $p_1 \cdot y_1 + p_2 \cdot y_2 + m_2 = M_2 + p_2 \cdot Q_2$. Solving those two maximization exercises by a Lagrange multiplier and applying the condition that the market remains in equilibrium, the competitive price vector $(\hat{p}_1, \hat{p}_2)$ is found as $\hat{p}_i = (\alpha_i/\alpha_m)(M_1 + M_2)/Q_i$ for $i = 1, 2$ (Chakrabarti and Chakrabarti 2009).

The outcomes of such a trading process are as follows.

(1) At optimal prices $(\hat{p}_1, \hat{p}_2)$, $m_1(t) + m_2(t) = m_1(t+1) + m_2(t+1)$, i.e. demand matches supply in all markets at the market-determined price in equilibrium. Since money is also treated as a commodity in this framework, its demand (i.e. the total amount of money held by the two persons after the trade) must be equal to what was supplied (i.e. the total amount of money held by them before the trade).

(2) If a restrictive assumption is made such that α_1 in the utility function can vary randomly over time with α_m remaining constant, it readily follows that α_2 also varies randomly over time with the restriction that the sum of α_1 and α_2 is a constant $(1-\alpha_m)$. Then in the derived money demand equations, if we assume α_m is λ and $\alpha_1/(\alpha_1+\alpha_2)$ is ϵ, it is found that the money evolution equations become

$$m_1(t+1) = \lambda m_1(t) + \epsilon(1-\lambda)(m_1(t) + m_2(t)),$$
$$m_2(t+1) = \lambda m_2(t) + (1-\epsilon)(1-\lambda)(m_1(t) + m_2(t)).$$

For a fixed value of λ, if α_1 (or α_2) is a random variable with uniform distribution over the domain $[0, 1-\lambda]$, then ϵ is also uniformly distributed over the domain $[0, 1]$. This limit corresponds to the Chakraborti and Chakrabarti (2000) model, discussed earlier.

(3) For the limiting value of α_m in the utility function (i.e. $\alpha_m \to 0$, which implies $\lambda \to 0$), the money transfer equation describing the random sharing of money without saving is obtained, which was studied by Dragulescu and Yakovenko (2000) mentioned earlier.

This actually demonstrates the equivalence of the two maximization principles of entropy (in physics) and utility (in economics), and is certainly noteworthy.

4. Agent-based modeling based on games

4.1. *Minority game models*

4.1.1. The El Farol Bar problem. Arthur (1994) introduced the 'El Farol Bar' problem as a paradigm of complex economic systems. In this problem, a population of agents have to decide whether to go to the bar opposite Santa Fe every Thursday night. Due to a limited number of seats, the bar cannot entertain more than X% of the population. If less than X% of the population go to the bar, the time spent in the bar is considered to be satisfying and it is better to attend the bar rather than stay at home. But if more than X% of the population go to the bar, then it is too crowded and people in the bar have an unsatisfying time. In this second case, staying at home is considered to be a better choice than attending the bar. Therefore, in order to optimize their own utility, each agent has to predict what everybody else will do.

In particular, Arthur was also interested in agents who have bounds on 'rationality', i.e. agents who:

- do not have perfect information about their environment, and, in general, they will only acquire information through interaction with the dynamically changing environment;
- do not have a perfect model of their environment;
- have limited computational power, so they cannot determine all the logical consequences of their knowledge; and
- have other resource limitations (e.g. memory).

In order to take these limitations into account, each agent is randomly given a fixed menu of models potentially suitable for predicting the number of people who will go to the bar given past data (e.g. the same as two weeks ago, the average of the past few weeks, etc.). Each week, each agent evaluates these models with respect to the past data. He chooses the one that was the best predictor for these data and then uses it to predict the number of people who will go to the bar this time. If this prediction is less than X, then the agent also decides to go to the bar. If its prediction is greater than X, the agent stays at home. Thus, in order to make decisions on whether or not to attend the bar, all the individuals are equipped with a certain number of 'strategies' that provide them with predictions concerning attendance in the bar next week based on the attendance in the past few weeks. As a result the number who go to the bar oscillates in an apparently random manner around the critical X% mark.

This was one of the first models that proceeded in a different way from traditional economics.

4.1.2. Basic minority game. Minority games (MGs) (Challet *et al.* 2004) refer to the multi-agent models of financial markets with the original formulation introduced by Challet and Zhang (1997), and all other variants (Lamper *et al.* 2002, Coolen 2005), most of which share the principal features that the models are repeated games and agents are inductive in nature. The original formulation of the minority game by Challet and Zhang (1997) is sometimes referred as the 'Original Minority Game' or the 'Basic Minority Game'.

The basic minority game consists of N (an odd natural number) agents, who choose between one of two decisions in each round of the game, using their own simple inductive strategies. The two decisions could be, for example, 'buying' or 'selling' commodities/assets, denoted by 0 or 1, at a given time t. An agent wins the game if he is one of the members of the minority group, and thus in each round, the minority group of agents win the game and rewards are given to those strategies that predict the winning side. All the agents have access to a finite amount of public information, which is a common bit-string 'memory' of the M most recent outcomes, composed of the winning sides in the past few rounds. Thus the agents with finite memory are said to exhibit 'bounded rationality' (Arthur 1994).

Consider, for example, memory $M=2$, then there are $P=2^M=4$ possible 'history' bit strings: 00, 01, 10 and 11. A 'strategy' consists of a response, i.e. 0 or 1, to each possible history bit string. Therefore, there are $G=2^P=2^{2^M}=16$ possible strategies that constitute the 'strategy space'. At the beginning of the game, each agent randomly picks k strategies, and after the game assigns one 'virtual' point to a strategy that would have predicted the correct outcome. The actual performance r of the player is measured by the number of times the player wins, and the strategy by which the player wins obtains a 'real' point. A record of the number of agents who have chosen a particular action, say 'selling' denoted by 1, $A_1(t)$, as a function of time is kept (see figure 14). The fluctuations in the behavior of $A_1(t)$ actually indicate the system's total utility. For example, we can have a situation where only one player is in the minority and all the other players lose. The other extreme case is when $(N-1)/2$ players are in the minority and $(N+1)/2$ players lose. The total utility of the system is obviously greater for the latter case and, from this perspective, the latter situation is more desirable. Therefore, the system is more efficient when there are smaller fluctuations around the mean than when the fluctuations are larger.

As in the El Farol bar problem, unlike most traditional economics models that assume agents are 'deductive' in nature, here also a 'trial-and-error' *inductive* thinking approach is implicitly implemented in the process of decision-making when agents make their choices in the games.

4.1.3. Evolutionary minority games. Challet and Zhang (1997, 1998) generalized the basic minority game mentioned above to include Darwinian selection: the worst player is replaced by a new one after some time steps, and the new player is a 'clone' of the best player, i.e. it inherits all the strategies but with the corresponding virtual capitals reset to zero (analogous to a new-born baby, and although having all the predispositions from the parents, it does not inherit their knowledge). To keep a certain diversity, they introduced the possibility of mutation when cloning. They allowed one of the strategies of the best player to be replaced by a new one. Since strategies are no longer just recycled among the players, the whole strategy phase space is available for selection. They expected this population to be capable of 'learning'

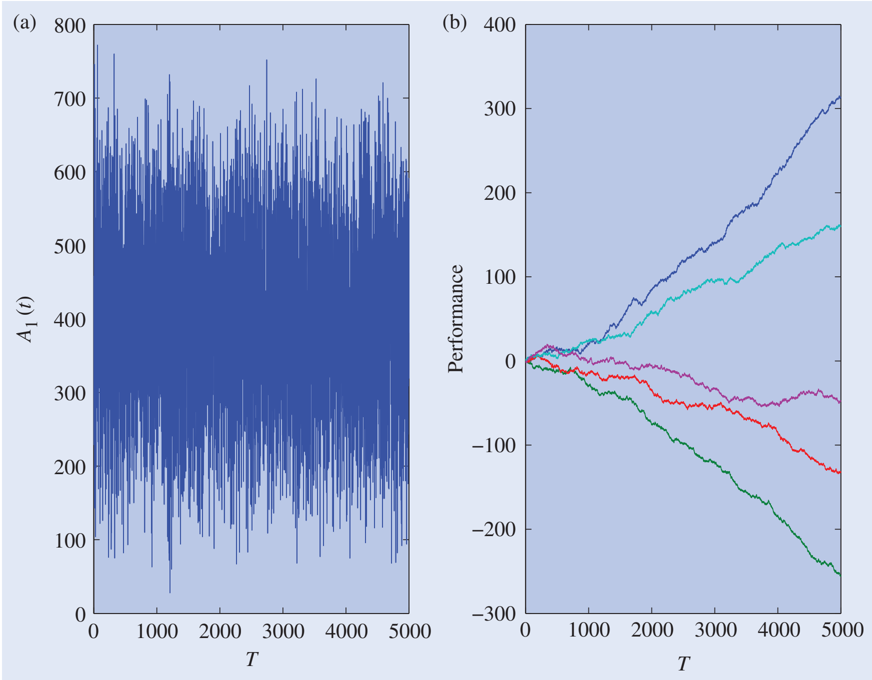

Figure 14. Attendance fluctuation and performances of players in the basic minority game. Plots of (a) attendance and (b) performance of the players (the five curves are: the best, the worst and three chosen randomly) for the basic minority game with $N = 801$, $M = 6$, $k = 10$ and $T = 5000$. Reproduced from Sysi-Aho *et al.* (2003b).

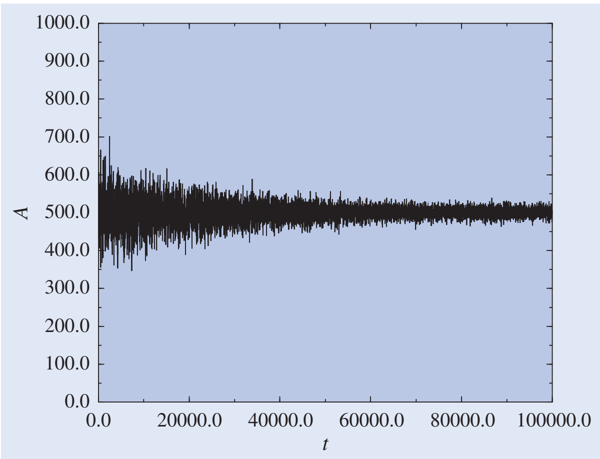

Figure 15. Temporal attendance A for the genetic approach showing a learning process. Reproduced from Challet and Zhang (1997).

since bad players are weeded out with time, and fighting is among the so-to-speak 'best' players. Indeed, from figure 15 they observed that learning emerged over time. Fluctuations are reduced and saturated, implying that the average gain for everybody is improved but never reaches the ideal limit.

Li *et al.* (2000a, b) also studied the minority game in the presence of 'evolution'. In particular, they examined the behavior in games in which the dimension of the strategy space, m, is the same for all agents and fixed for all time. They found that, for all values of m, not too large, evolution results in a substantial improvement in overall system performance. They also showed that, after evolution, the results obeyed a scaling relation among games played with different values of m and different numbers of agents, analogous to that found in the non-evolutionary, adaptive games (see remarks in section 4.1.5). The best system performance still occurred, for a given number of agents, at m_c, the same value of the dimension of the strategy space as in the non-evolutionary case, but system performance was nearly an order of magnitude better than the non-evolutionary result. For $m < m_c$, the system evolved to states in which average agent wealth was better than in the random choice game. As m became large, overall systems performance approached that of the random choice game.

Li *et al.* (2000a, b) continued the study of evolution in minority games by examining games in which agents with poorly performing strategies can trade in their strategies for new ones from a different strategy space, which means allowing for strategies that use information from different numbers of time lags, m. They found in all the games that, after evolution, wealth per agent is high for agents with strategies drawn from small strategy spaces (small m), and low for agents with strategies drawn from large strategy spaces (large m). In the game played with N agents, wealth per agent as a function of m was very nearly a step function. The transition was found to be at $m = m_t$, where $m_t \simeq m_c - 1$, and m_c is the critical value of m at which N agents playing the game with a fixed strategy space (fixed m) have the best emergent coordination and the best utilization of resources. They also found that, overall, system-wide utilization of resources is independent of N. Furthermore, although overall system-wide utilization of resources after evolution varied somewhat depending on

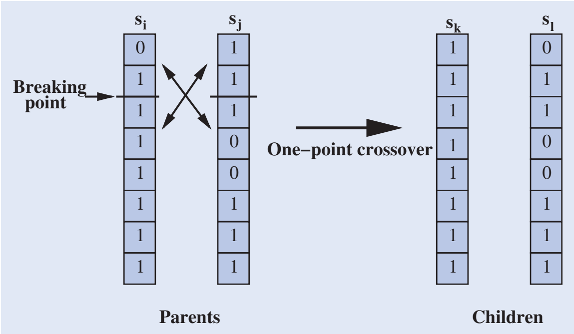

Figure 16. Schematic diagram illustrating the mechanism of one-point genetic crossover for producing new strategies. Strategies s_i and s_j are the parents. We choose the breaking point randomly and using this one-point genetic crossover the children s_k and s_l are produced and substitute for the parents. Reproduced from Sysi-Aho *et al.* (2003b).

certain aspects of the evolutionary dynamics, in the best cases utilization of resources was on the order of the best results achieved in evolutionary games with fixed strategy spaces.

4.1.4. Adaptive minority games. Sysi-Aho *et al.* (2003a, b, c, 2004) presented a simple modification of the basic minority game where the players modify their strategies periodically after every time interval τ, depending on their performance: if a player finds that he is among the fraction n (where $0 < n < 1$) who are the worst performing players, he adapts himself and modifies his strategies. They proposed that the agents use hybridized one-point genetic crossover mechanism (as shown in figure 16), inspired by genetic evolution in biology, to modify the strategies and replace the bad strategies. They studied the performances of the agents under different conditions and investigated how they adapted themselves in order to survive or to be the best by finding new strategies using the highly effective mechanism.

They also studied the measure of total utility of the system $U(x_t)$, which is the number of players in the minority group; the total utility of the system is maximum $U_{\max}$ if the highest number of winning players is $(N-1)/2$. The system is more efficient when the deviations from the maximum total utility $U_{\max}$ are smaller or, in other words, the fluctuations in $A_1(t)$ around the mean become smaller. Interestingly, the fluctuations disappear totally and the system stabilizes to a state where the total utility of the system is at a maximum, since, at each time step, the largest number of players win the game (see figure 17). As expected, the behavior depends on the parameter values for the system (Sysi-Aho *et al.* 2003b, 2004). They used the utility function to study the efficiency and dynamics of the game, as shown in figure 18.

If the parents are chosen randomly from the pool of strategies, then the mechanism represents a 'one-point genetic crossover' and if the parents are the best strategies, then the mechanism represents a 'hybridized genetic crossover'. The children may replace parents or the two worst strategies and accordingly four different interesting cases arise: (a) one-point genetic crossover with parents 'killed', i.e. the parents are replaced by the children; (b) one-point genetic crossover with parents 'saved', i.e. the two worst strategies are replaced by the children but the parents are retained; (c) hybridized genetic crossover with parents 'killed'; and (d) hybridized genetic crossover with parents 'saved'.

In order to determine which mechanism is the most efficient, we performed a comparative study of the four cases mentioned above. We plot the attendance as a function of time for the different mechanisms in figure 19. In figure 20 we show the total utility of the system in each of the cases (a)–(d), where we have plotted the results of the average over 100 runs and each point in the utility curve represents a time average taken over a bin of length 50 time-steps. The simulation time is doubled from those in figure 19 in order to expose the asymptotic behavior better. On the basis of figures 19 and 20, we find that case (d) is the most efficient.

In order to investigate what happens at the level of the individual agent, we created a competitive surrounding 'test' situation where, after $T = 3120$ time-steps, six players begin to adapt and modify their strategies such that three are using the hybridized genetic crossover mechanism and the other three the one-point genetic crossover mechanism, where children replace the parents. The rest of the players play the basic minority game. In this case it turns out that, in the end, the best players are those who use the hybridized mechanism, the second best are those using the one-point mechanism, and the bad players are those who do not adapt at all. In addition, it turns out that the competition amongst the players who adapt using the hybridized genetic crossover mechanism is severe.

It should be noted that the mechanism of evolution of strategies is considerably different from earlier attempts such as those of Challet and Zhang (1997) and Li *et al.* (2000a, b). This is because, in this mechanism, the strategies are changed by the agents themselves and even though the strategy space evolves continuously, its size and dimensionality remain the same.

Due to the simplicity of these models (Sysi-Aho *et al.* 2003a, b, c, 2004) a lot of freedom is found in modifying the models to make the situations more realistic and applicable to many real dynamical systems, and not only financial markets. Many details of the model can be fine-tuned to imitate real markets or the behavior of other complex systems. Many other sophisticated models based on these games can be set up and implemented and show great potential with respect to the commonly adopted statistical techniques in analyses of financial markets.

4.1.5. Remarks. For modeling purposes, the minority game models were meant to serve as a class of simple models that could produce some macroscopic features observed in real financial markets, including the fat-tail price return distribution and volatility clustering

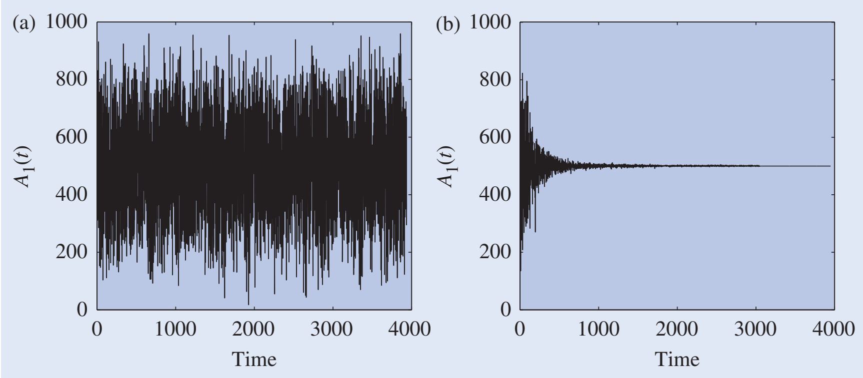

Figure 17. Plot showing the time variations of the number of players A_1 who choose action 1, with parameters $N=1001$, $m=5$, $s=10$ and $t=4000$ for (a) the basic minority game and (b) the adaptive game, where $\tau=25$ and $n=0.6$. Reproduced from Sysi-Aho *et al.* (2003a).

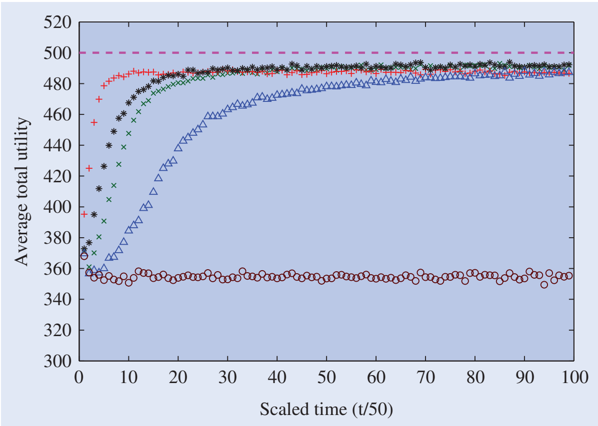

Figure 18. Plot showing the variation of the total utility of the system with time for the basic minority game for $N=1001$, $m=5$, $s=10$ and $t=5000$, and the adaptive game for the same parameters but different values of τ and n. Each point represents a time average of the total utility for separate bins of size 50 time-steps of the game. The maximum total utility ($=(N-1)/2$) is shown as the dashed line. (○) Data for the basic minority game. (+) Data for $\tau=10$ and $n=0.6$. (∗) Data for $\tau=50$ and $n=0.6$. (×) Data for $\tau=10$ and $n=0.2$. (△) Data for $\tau=50$ and $n=0.2$. The ensemble average over 70 different samples was taken in each case. Reproduced from Sysi-Aho *et al.* (2003a).

(Challet *et al.* 2004, Coolen 2005). Despite the hectic activity (Challet and Zhang 1998, Challet *et al.* 2000), they have failed to capture or reproduce the most important stylized facts of real markets. However, in the physicists' community, they have become an interesting and established class of models concerning the physics of disordered systems (Cavagna *et al.* 1999, Challet *et al.* 2000), providing much physical insight (Savit *et al.* 1999, Martino *et al.* 2004). Since, in the BMG model, a Hamiltonian function can be defined and analytic solutions developed in certain regimes of the model, the model was viewed from a more physical perspective. In fact, it is characterized by a clear two-phase structure with very different collective behavior in the two phases, as in many known conventional physical systems (Cavagna *et al.* 1999, Savit *et al.* 1999).

Savit *et al.* (1999) first found that the macroscopic behavior of the system does not depend independently on the parameters N and M, but instead depends on the ratio

$$\alpha \equiv \frac{2^M}{N} = \frac{P}{N}, \tag{35}$$

which serves as the most important control parameter in the game. The variance in the attendance (see also Sysi-Aho *et al.* (2003c)), or volatility σ^2/N, for different values of N and M depends only on the ratio α. Figure 21 shows a plot of σ^2/N versus the control parameter α, where the data collapse of σ^2/N for different values of N and M is clearly evident. The dotted line in figure 21 corresponds to the 'coin-toss' limit (random choice or pure chance limit), in which agents play by simply making random decisions (by coin-tossing) in every round of the game. This value of σ^2/N in the coin-toss limit can be obtained by simply assuming a binomial distribution of the agents' binary actions, with probability 0.5, such that $\sigma^2/N=0.5(1-0.5)\cdot 4=1$. When α is small, the value of σ^2/N of the game is larger than the coin-toss limit, which implies that the collective behaviors of the agents are worse than the random choices. In the early literature, this was popularly called the *worse-than-random* regime. When α increases, the value of σ^2/N decreases and enters a region where agents are performing better than the random choices, which was popularly called the *better-than-random* regime. The value of σ^2/N reaches a minimum value that is substantially smaller than the coin-toss limit. When α increases further, the value of σ^2/N again increases and approaches the coin-toss limit. This allowed one to identify two phases in the minority game separated by the minimum value of σ^2/N in the graph. The value of α where the rescaled volatility was minimum was denoted α_c, which represented the phase transition point; α_c has been shown to have a value of 0.3374... (for $k=2$) by analytical calculations (Challet *et al.* 2000).

Besides these collective behaviors, physicists also became interested in the dynamics of the games such as the crowd versus anti-crowd movement of agents, periodic attractors, etc. (Johnson *et al.* 1999a, b, Hart *et al.* 2001). In this way, the minority games serve as a useful tool and provide a new direction for physicists in viewing and analysing the underlying dynamics of complex evolving systems such as financial markets.

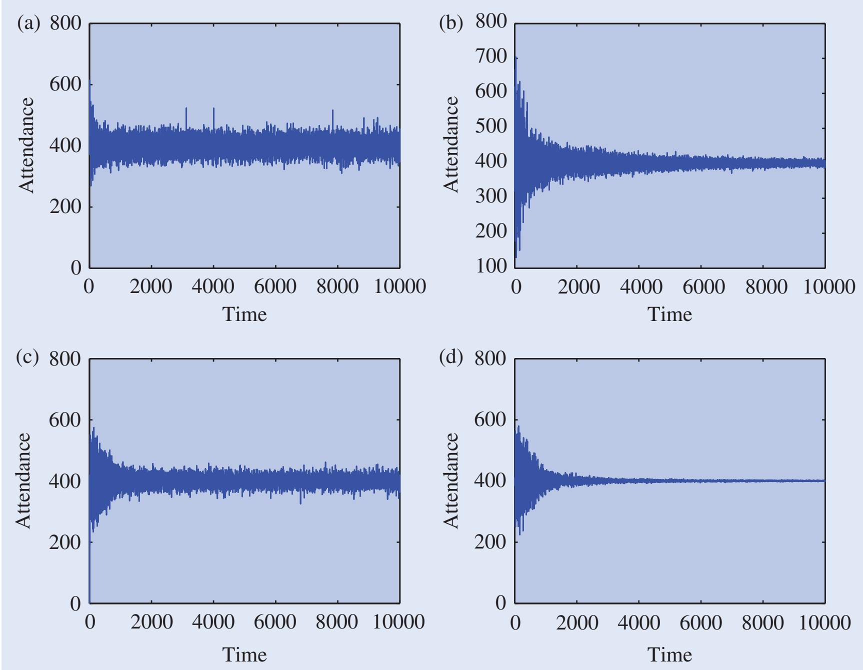

Figure 19. Plots of the attendance when choosing parents randomly (a, b), and using the best parents in a players' pool (c, d). In (a) and (c), parents are replaced by children, and in (b) and (d), children replace the two worst strategies. Simulations were performed with $N=801$, $M=6$, $k=16$, $t=40$, $n=0.4$ and $T=10{,}000$. Reproduced from Sysi-Aho *et al.* (2003b).

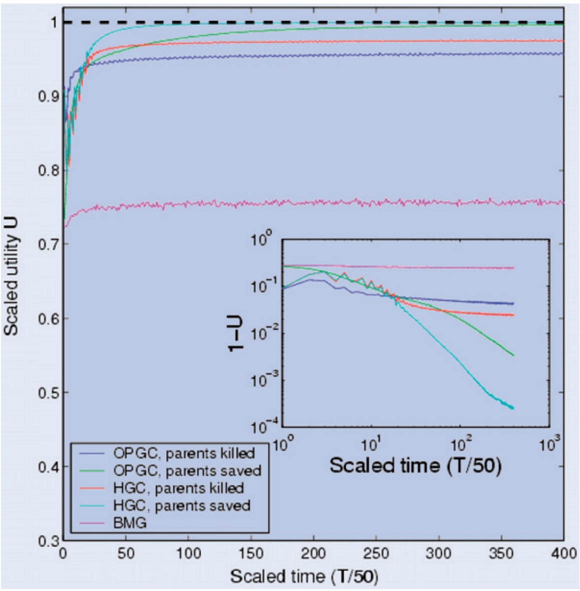

Figure 20. Plots of the scaled utilities of the four different mechanisms compared with that of the basic minority game. Each curve represents an ensemble average over 100 runs and each point on a curve is a time average over a bin of length 50 time-steps. Inset: the quantity $(1-U)$ is plotted versus the scaled time on a double logarithmic scale. Simulations were performed with $N=801$, $M=6$, $k=16$, $t=40$, $n=0.4$ and $T=20{,}000$. Adapted from Sysi-Aho *et al.* (2003b).

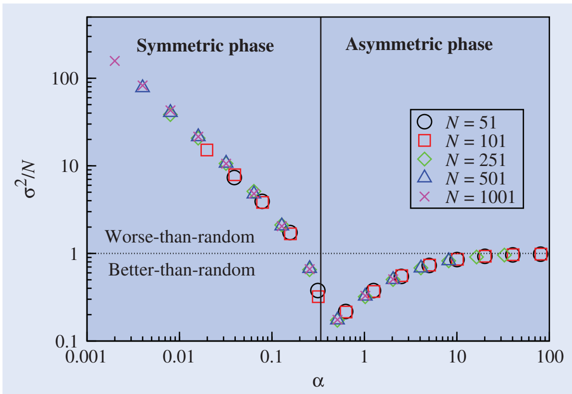

Figure 21. Simulation results for the variance in attendance σ^2/N as a function of the control parameter $\alpha=2^M/N$ for games with $k=2$ strategies for each agent, with the ensemble averaged over 100 sample runs. The dotted line shows the value of the volatility in the random choice limit. The solid line shows the critical value of $\alpha=\alpha_c\approx 0.3374$. Reproduced from Yeung and Zhang (arxiv:0811.1479).

4.2. *The Kolkata Paise Restaurant (KPR) problem*

The KPR problem (Chakrabarti *et al.* 2009, Ghosh and Chakrabarti 2009, Ghosh *et al.* 2010a, b) is a repeated game played between a large number N of agents having no interaction amongst themselves. In the KPR problem, prospective customers (agents) choose from N restaurants each evening simultaneously (in parallel decision mode); N is fixed. Each restaurant has the same price for a meal but a different rank (agreed upon by all customers) and can serve only one customer any evening. Information regarding the customer distributions for earlier evenings is available to everyone. Each customer's objective is to go to the restaurant with the highest possible rank while avoiding the crowd so as to be able to get dinner there. If more than one customer arrives at any restaurant on any evening, one of them is randomly chosen (each of them are treated anonymously) and is served. The rest do not get dinner that evening.

In Kolkata, there were very cheap and fixed-rate 'Paise Restaurants' that were popular among the daily laborers in the city. During lunch hours, the laborers used to walk (to save the transport costs) to one of these restaurants and would miss lunch if they got to a restaurant where there were too many customers. Walking down to the next restaurant would mean failing to report back to work on time! Paise is the smallest Indian coin and there were indeed some well-known rankings of these restaurants, as some of them would offer tastier items than the others. A more general example of such a problem would be when society provides hospitals (and beds) in every locality but the local patients go to hospitals of (commonly perceived) better rank elsewhere, thereby competing with the local patients of those hospitals. The unavailability of treatment in time may be considered as a lack of service for those people and consequently as a (social) wastage of service by those unattended hospitals.

A dictator's solution to the KPR problem is the following: the dictator asks everyone to form a queue and then assigns each one a restaurant with rank matching the sequence of the person in the queue on the first evening. Then each person is told to go to the next ranked restaurant on the following evening (for the person in the last ranked restaurant this means going to the first ranked restaurant). This shift then proceeds continuously for successive evenings. This is clearly one of the most efficient solutions (with a utilization fraction $\bar{f}$ of the services by the restaurants equal to unity) and the system arrives at this solution immediately (from the first evening). However, in reality, this cannot be the true solution of the KPR problem, where each agent decides on his own (in parallel or democratically) every evening, based on complete information about past events. In this game, the customers try to evolve a learning strategy to eventually obtain dinner at the best possible ranked restaurant, avoiding the crowd. It can be seen that the evolution of these strategies takes a considerable time to converge and even then the eventual utilization fraction $\bar{f}$ is far less than unity.

Let the symmetric stochastic strategy chosen by each agent be such that, at any time t, the probability $p_k(t)$ of arriving at the kth ranked restaurant is given by

$$\begin{aligned} p_k(t) &= \frac{1}{z}\left[k^\alpha \exp\left(-\frac{n_k(t-1)}{T}\right)\right], \\ z &= \sum_{k=1}^{N}\left[k^\alpha \exp\left(-\frac{n_k(t-1)}{T}\right)\right], \end{aligned} \tag{36}$$

where $n_k(t)$ denotes the number of agents arriving at the kth ranked restaurant in period t, $T>0$ is a scaling factor and $\alpha \geq 0$ is an exponent.

For any natural number α and $T\to\infty$, an agent goes to the kth ranked restaurant with probability $p_k(t)=k^\alpha/\sum k^\alpha$, which means that, in the limit $T\to\infty$ in (36), $p_k(t)=k^\alpha/\sum k^\alpha$. If an agent selects any restaurant with equal probability p, then the probability that a single restaurant is chosen by m agents is given by

$$\Delta(m) = \binom{N}{m} p^m (1-p)^{N-m}. \tag{37}$$

Therefore, the probability that a restaurant with rank k is not chosen by any of the agents will be given by

$$\begin{aligned} \Delta_k(m=0) &= \binom{N}{0}(1-p_k)^N, \quad p_k = \frac{k^\alpha}{\sum k^\alpha} \\ &\simeq \exp\left(\frac{-k^\alpha N}{\widetilde{N}}\right) \quad \text{as } N\to\infty, \end{aligned} \tag{38}$$

where $\widetilde{N} = \sum_{k=1}^{N} k^\alpha \simeq \int_0^N k^\alpha \, \mathrm{d}k = N^{\alpha+1}/(\alpha+1)$. Hence,

$$\Delta_k(m=0) = \exp\left(-\frac{k^\alpha(\alpha+1)}{N^\alpha}\right). \tag{39}$$

Therefore, the average fraction of agents receiving dinner in the kth ranked restaurant is given by

$$\bar{f}_k = 1 - \Delta_k(m=0). \tag{40}$$

Naturally, for $\alpha=0$, the problem corresponding to random choice $\bar{f}_k = 1 - \mathrm{e}^{-1}$ gives $\bar{f} = \sum \bar{f}_k/N \simeq 0.63$ and for $\alpha=1$, $\bar{f}_k = 1-\mathrm{e}^{-2k/N}$ gives $\bar{f} = \sum \bar{f}_k/N \simeq 0.58$.

In summary, in the KPR problem the decision made by each agent on each evening t is independent and is based on information concerning the rank k of the restaurants and their occupancy given by $n_k(t-1), \ldots, n_k(0)$. For several stochastic strategies, only $n_k(t-1)$ is utilized and each agent chooses the kth ranked restaurant with probability $p_k(t)$ given by equation (36). The utilization fraction f_k of the kth ranked restaurant on every evening was studied and their average (over k) distributions $D(f)$ were studied numerically, as well as analytically, and it was found (Chakrabarti *et al.* 2009, Ghosh and Chakrabarti 2009, Ghosh *et al.* 2010a) that their distributions are Gaussian with the most probable utilization fraction $\bar{f}\simeq 0.63$, 0.58 and 0.46 for cases with $\alpha=0$, $T\to\infty$; $\alpha=1$, $T\to\infty$; and $\alpha=0$, $T\to 0$, respectively. For the stochastic crowd-avoiding strategy discussed by Ghosh *et al.* (2010b), where $p_k(t+1)=1/n_k(t)$ for $k=k_0$, the restaurant visited by the agent the last evening, and $=1/(N-1)$ for all other restaurants ($k\neq k_0$), one obtains the best utilization fraction $\bar{f}\simeq 0.8$, and the analytical estimates for $\bar{f}$ in these limits agree very well with the numerical observations. Also, the time required to converge to the above value of $\bar{f}$ is independent of N.

The KPR problem has similarities to the Minority Game Problem (Arthur 1994, Challet *et al.* 2004) as, in both the games, herding behavior is punished and diversity is encouraged. Also, both involve agents learning from past successes, etc. Of course, KPR has some simple exact solution limits, a few of which are discussed here. The real challenge is, of course, to design algorithms for learning mixed strategies (e.g., from the pool discussed here) by the agents so that the fair social norm eventually emerges (in N^0 or ln N order time) even when everyone decides on the basis of their own information independently. As we have seen, some naive strategies give better values of $\bar{f}$ than most of the 'smarter' strategies such as strict crowd-avoiding strategies, etc. This observation in fact compares well with earlier observations of minority games (see, e.g., Satinover and Sornette (2007)).

It may be noted that all the stochastic strategies, being parallel in computational mode, have the advantage that they converge to a solution in smaller time steps ($\sim N^0$ or $\ln N$), whereas for deterministic strategies the convergence time is typically of order N, which renders such strategies useless in the truly macroscopic ($N \to \infty$) limits. However, deterministic strategies are useful when N is small and rational agents can design appropriate punishment schemes for the deviators (Kandori 2008).

Study of the KPR problem shows that while a dictated solution leads to one of the best possible solutions to the problem, with each agent obtaining his dinner at the best ranked restaurant in a period of N evenings, and with a best possible value of $\bar{f}$ (=1) starting from the first evening, the parallel decision strategies (employing evolving algorithms used by the agents and past information, for example $n(t)$), which are necessarily parallel among the agents and stochastic (as in a democracy), are less efficient ($\bar{f} \ll 1$; the best is discussed by Ghosh *et al.* (2010b), giving $\bar{f} \simeq 0.8$ only). Note here that the time required is not dependent on N. We also note that most of the 'smarter' strategies lead to much lower efficiency.

5. Conclusions and outlook

Agent-based models of order books are a good example of the interactions between ideas and methods that are usually linked either to Economics and Finance (microstructure of markets, agent interaction) or to Physics (reaction–diffusion processes, the deposition–evaporation process, kinetic theory of gases). Today, the existing models exhibit a trade-off between 'realism' and calibration of the mechanisms and processes (empirical models such as that of Mike and Farmer (2008)) and the explanatory power of simple observed behavior (see Cont and Bouchaud (2000) and Cont (2007), for example). In the first case, some of the 'stylized facts' may be reproduced, but using empirical processes that may not be linked to any behavior observed in the market. In the second case, these are only toy models that cannot be calibrated to data. The mixing of many features, as in Lux and Marchesi (2000) and as is usually the case in behavioral finance, leads to poorly tractable models where the sensitivity to one parameter is hardly understandable. Therefore, no empirical model can tackle properly empirical facts such as volatility clustering. Importing toy model features to explain volatility clustering or market interactions in order book models has yet to be determined. Finally, let us also note that, to our knowledge, no agent-based model of order books deals with the multidimensional case. Implementing agents trading several assets simultaneously in a way that reproduces empirical observations on correlation and dependence remains an open challenge.

We believe that this type of modeling is crucial for future developments in finance. The financial crisis that occurred in 2007–2008 is expected to create a shock in classic modeling in Economics and Finance. Many scientists have expressed their views on this subject (e.g. Bouchaud (2008), Farmer and Foley (2009) and Lux and Westerhoff (2009)) and we also believe that the agent-based models presented here will be at the core of future modeling. As examples, we mention Iori *et al.* (2006), who model the interbank market and investigate systemic risk, Thurner *et al.* (2009), who investigate the effects of the use of leverage and margin calls on the stability of a market, and Yakovenko and Rosser (2009), who provide a brief overview of the study of wealth distributions and inequalities. No doubt these will be followed by many other contributions.

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References

Alfi, V., Cristelli, M., Pietronero, L. and Zaccaria, A., Minimal agent based model for financial markets I. *Eur. Phys. J. B*, 2009a, **67**, 385–397.

Alfi, V., Cristelli, M., Pietronero, L. and Zaccaria, A., Minimal agent based model for financial markets II. *Eur. Phys. J. B*, 2009b, **67**, 399–417.

Angle, J., The surplus theory of social stratification and the size distribution of personal wealth, in *Proceedings of the American Social Statistical Association, Social Statistics Section*, Alexandria, VA, 1983, p. 395.

Angle, J., The surplus theory of social stratification and the size distribution of personal wealth. *Social Forces*, 1986, **65**, 293–326.

Angle, J., The Statistical signature of pervasive competition on wage and salary incomes. *J. Math. Sociol.*, 2002, **26**, 217–270.

Angle, J., The inequality process as a wealth maximizing process. *Physica A*, 2006, **367**, 388–414.

Aoyama, H., Souma, W. and Fujiwara, Y., Growth and fluctuations of personal and company's income. *Physica A*, 2003, **324**, 352–358.

Arthur, W.B., Inductive reasoning and bounded rationality. *Am. Econ. Rev.*, 1994, **84**, 406–411.

Ausloos, M. and Pekalski, A., Model of wealth and goods dynamics in a closed market. *Physica A*, 2007, **373**, 560–568.

Bak, P., Paczuski, M. and Shubik, M., Price variations in a stock market with many agents. *Physica A*, 1997, **246**, 430–453.

Benhabib, J. and Bisin, A., The distribution of wealth and fiscal policy in economies with finitely lived agents. National Bureau of Economic Research Working Paper Series, No. 14730, 2009.

Bennati, E., *La Simulazione Statistica Nell'analisi Della Distribuzione del Reddito: Modelli Realistici e Metodo di Monte Carlo*, 1988a (ETS Editrice: Pisa).

Bennati, E., Un metodo di simulazione statistica nell'analisi della distribuzione del reddito. *Riv. Int. Sci. Econ. Commerc.*, 1988b, **35**, 735–756.

Bennati, E., Il metodo Monte Carlo nell'analisi economica, in *Proceedings of the Rassegna di Lavori dell'ISCO X*, Vol. 10, 1993, pp. 31–79.

Biais, B., Hillion, P. and Spatt, C., An empirical analysis of the limit order book and the order flow in the Paris Bourse. *J. Finance*, 1995, 1655–1689.

Bouchaud, J.-P., Economics needs a scientific revolution. *Nature*, 2008, **455**, 1181.

Bouchaud, J.-P. and Mezard, M., Wealth condensation in a simple model of economy. *Physica A*, 2000, **282**, 536–545.

Bouchaud, J.-P., Mézard, M. and Potters, M., Statistical properties of stock order books: empirical results and models. *Quant. Finance*, 2002, **2**, 251–256.

Bouchaud, J.-P., Farmer, J.D. and Lillo, F., How markets slowly digest changes in supply and demand. In *Handbook of Financial Markets: Dynamics and Evolution*, edited by T. Hens and K.R. Schenk-Hopp, pp. 57–160, 2009 (North-Holland: San Diego).

Burda, Z., Jurkiewics, J. and Nowak, M.A., Is Econophysics a solid science? *Acta Phys. Polon. B*, 2003, **34**, 87–131.

Case, K.E. and Fair, R.C., *Principles of Economics*, International edition, 2008 (Pearson Education Inc.: Upper Saddle River, NJ).

Cavagna, A., Garrahan, J.P., Giardina, I. and Sherrington, D., Thermal model for adaptive competition in a market. *Phys. Rev. Lett.*, 1999, **21**, 4429–4432.

Chakraborti, A., Distributions of money in model markets of economy. *Int. J. Mod. Phys. C*, 2002, **13**, 1315–1322.

Chakraborti, A. and Chakrabarti, B.K., Statistical mechanics of money: how saving propensity affects its distribution. *Eur. Phys. J. B*, 2000, **17**, 167–170.

Chakrabarti, A.S. and Chakrabarti, B.K., Microeconomics of the ideal gas like market models. *Physica A*, 2009, **388**, 4151–4158.

Chakrabarti, A.S., Chakrabarti, B.K., Chatterjee, A. and Mitra, M., The Kolkata Paise Restaurant problem and resource utilization. *Physica A*, 2009, **388**, 2420–2426.

Chakrabarti, B.K. and Chatterjee, A., Ideal gas-like distributions in economics: effects of saving propensity. In *Proceedings of the Application of Econophysics*, edited by H. Takayasu, pp. 280–285, 2003 (Springer: Tokyo).

Chakraborti, A. and Patriarca, M., Gamma-distribution and wealth inequality. *Pramana J. Phys.*, 2008, **71**, 233–243.

Chakraborti, A. and Patriarca, M., Variational principle for the Pareto power law. *Phys. Rev. Lett.*, 2009, **103**, 228701-1–228701-4.

Challet, D., Marsili, M. and Zhang, Y.C., *Minority Games*, 2004 (Oxford University Press: Oxford).

Challet, D., Marsili, M. and Zecchina, R., Statistical mechanics of systems with heterogeneous agents: Minority games. *Phys. Rev. Lett.*, 2000, **84**, 1824–1827.

Challet, D. and Stinchcombe, R., Analyzing and modeling 1 + 1d markets. *Physica A*, 2001, **300**, 285–299.

Challet, D. and Zhang, Y.C., Emergence of cooperation and organization in an evolutionary game. *Physica A*, 1997, **246**, 407–418.

Challet, D. and Zhang, Y.C., On the minority game: analytical and numerical studies. *Physica A*, 1998, **256**, 514–532.

Champernowne, D.G., A model of income distribution. *Econ. J.*, 1953, **63**, 318–351.

Chatterjee, A. and Chakrabarti, B.K., Kinetic exchange models for income and wealth distributions. *Eur. Phys. J. B*, 2007, **60**, 135–149.

Chatterjee, A., Chakrabarti, B.K. and Stinchcombe, R.B., Master equation for a kinetic model of trading market and its analytic solution. *Phys. Rev. E*, 2005a, **72**, 026126-1–026126-4.

Chatterjee, A., Yarlagadda, S. and Chakrabarti, B.K., editors, *Econophysics of Wealth Distributions*, 2005b (Springer: Milan).

Chatterjee, A., Chakrabarti, B.K. and Manna, S.S., Money in gas-like markets: Gibbs and Pareto laws. *Phys. Scripta*, 2003, **T106**, 36–38.

Chatterjee, A., Chakrabarti, B.K. and Manna, S.S., Pareto law in a kinetic model of market with random saving propensity. *Physica A*, 2004, **335**, 155–163.

Chiarella, C. and Iori, G., A simulation analysis of the microstructure of double auction markets. *Quant. Finance*, 2002, **2**, 346–353.

Cliff, D. and Bruten, J., Zero is not enough: on the lower limit of agent intelligence for continuous double auction markets. Technical Report HPL-97-141, Hewlett-Packard Laboratories, Bristol, UK, 1997.

Cohen, M., Report of the special study of the securities markets of the Securities and Exchanges Commission. Technical Report, U.S. Congress, 88th Congr., 1st session, H. Document 95, 1963a.

Cohen, M.H., Reflections on the special study of the Securities Markets. Technical Report, Speech at the Practising Law Intitute, New York, 1963b.

Cont, R., Volatility clustering in financial markets: empirical facts and agent-based models. In *Long Memory in Economics*, edited by G. Teyssière and A.P. Kirman, pp. 289–309, 2007 (Springer: Berlin).

Cont, R. and Bouchaud, J.P., Herd behavior and aggregate fluctuations in financial markets. *Macroecon. Dynam.*, 2000, **4**, 170–196.

Cont, R., Stoikov, S. and Talreja, R., A stochastic model for order book dynamics. SSRN eLibrary, 2008.

Coolen, A., *The Mathematical Theory of Minority Games: Statistical Mechanics of Interacting Agents*, 2005 (Oxford University Press: Oxford).

Cordier, S., Pareschi, L. and Toscani, G., On a kinetic model for a simple market economy. *J. Statist. Phys.*, 2005, **120**, 253–277.

Das, A. and Yarlagadda, S., A distribution function analysis of wealth distribution, arxiv.org:cond-mat/0310343, 2003.

Das, A. and Yarlagadda, S., An analytic treatment of the Gibbs–Pareto behavior in wealth distribution. *Physica A*, 2005, **353**, 529–538.

De Martino, A.D., Giardina, I., Tedeschi, A. and Marsili, M., Generalized minority games with adaptive trend-followers and contrarians. *Phys. Rev. E*, 2004, **70**, 025104-1–025104-4.

Dragulescu, A. and Yakovenko, V.M., Statistical mechanics of money. *Eur. Phys. J. B*, 2000, **17**, 723–729.

Dragulescu, A. and Yakovenko, V.M., Evidence for the exponential distribution of income in the USA. *Eur. Phys. J. B*, 2001a, **20**, 585–589.

Dragulescu, A. and Yakovenko, V.M., Exponential and power-law probability distributions of wealth and income in the United Kingdom and the United States. *Physica A*, 2001b, **299**, 213–221.

Düring, B., Matthes, D. and Toscani, G., Kinetic equations modelling wealth redistribution: a comparison of approaches. *Phys. Rev. E*, 2008, **78**, 056103-1–056103-12.

Düring, B. and Toscani, G., Hydrodynamics from kinetic models of conservative economies. *Physica A*, 2007, **384**, 493–506.

Erdos, P. and Renyi, A., On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci.*, 1960, **5**, 17–61.

Farmer, J.D. and Foley, D., The economy needs agent-based modelling. *Nature*, 2009, **460**, 685–686.

Ferrero, J.C., The statistical distribution of money and the rate of money transference. *Physica A*, 2004, **341**, 575–585.

Fujiwara, Y., Souma, W., Aoyama, H., Kaizoji, T. and Aoki, M., Growth and fluctuations of personal income. *Physica A*, 2003, **321**, 598–604.

Gabaix, X., Zipf's law For cities: an explanation. *Q. J. Econ.*, 1999, **114**, 739–767.

Garibaldi, U., Scalas, E. and Viarengo, P., Statistical equilibrium in simple exchange games II. The redistribution game. *Eur. Phys. J. B*, 2007, **60**, 241–246.

Garman, M.B., Market microstructure. *J. Financial Econ.*, 1976, **3**, 257–275.

Ghosh, A. and Chakrabarti, B.K., Mathematica demonstration of the Kolkata Paise Restaurant (KPR) problem. 2009. Available online at: http://demonstrations.wolfram.com/

Ghosh, A., Chakrabarti, A.S. and Chakrabarti, B.K., Kolkata Paise Restaurant problem in some uniform learning strategy limits. In *Econophysics & Economics of Games, Social Choices & Quantitative Techniques*, edited by B. Basu, B.K. Chakrabarti, S.R. Chakravarty and K. Gangopadhyay, pp. 3–9, 2010a (Springer: Milan).

Ghosh, A., Chatterjee, A., Mitra, M. and Chakrabarti, B.K., Statistics of the Kolkata Paise Restaurant problem. *New J. Phys.*, 2010b, **12**, 075033-1–075033-10.

Gibrat, R., *Les Inégalités Economiques*, 1931 (Sirey: Paris).

Gode, D.K. and Sunder, S., Allocative efficiency of markets with zero-intelligence traders: market as a partial substitute for individual rationality. *J. Polit. Econ.*, 1993, **101**, 119–137.

Gu, G. and Zhou, W., Emergence of long memory in stock volatility from a modified Mike–Farmer model. *Europhys. Lett.*, 2009, **86**, 48002-1–48002-6.

Gupta, A.K., Money exchange model and a general outlook. *Physica A*, 2006, **359**, 634–640.

Gupta, A.K., Relaxation in the wealth exchange models. *Physica A*, 2008, **387**, 6819–6824.

Hakansson, N., Beja, A. and Kale, J., On the feasibility of automated market making by a programmed specialist. *J. Finance*, 1985, **XL**, 1–20.

Hart, M., Jefferies, P., Hui, P. and Johnson, N., Crowd–anticrowd theory of multi-agent market games. *Eur. Phys. J. B*, 2001, **20**, 547–550.

Hasbrouck, J., *Empirical Market Microstructure: The Institutions, Economics, and Econometrics of Securities Trading*, 2007 (Oxford University Press: New York).

Hayes, B., Follow the money. *Am. Sci.*, 2002, **90**, 400–405.

Heinsalu, E., Patriarca, M. and Marchesoni, F., Stochastic resonance in bistable confining potentials: On the role of confinement. *Eur. Phys. J. B.*, 2009, **69**, 19–22.

Iglesias, J.R., Goncalves, S., Abramsonb, G. and Vega, J.L., Correlation between risk aversion and wealth distribution. *Physica A*, 2004, **342**, 186–192.

Iglesias, J.R., Goncalves, S., Pianegonda, S., Vega, J.L. and Abramson, G., Wealth redistribution in our small world. *Physica A*, 2003, **327**, 12–17.

Iori, G., Jafarey, S. and Padilla, F.G., Systemic risk on the interbank market. *J. Econ. Behav. Organiz.*, 2006, **61**, 525–542.

Ispolatov, S., Krapivsky, P.L. and Redner, S., Wealth distributions in asset exchange models. *Eur. Phys. J. B*, 1998, **2**, 267–276.

Johnson, N.F., Hui, P.M., Jonson, R. and Lo, T.S., Self-organized segregation within an evolving population. *Phys. Rev. Lett.*, 1999a, **82**, 3360–3363.

Johnson, N.F., Hart, M. and Hui, P.M., Crowd effects and volatility in markets with competing agents. *Physica A*, 1999b, **269**, 1–8.

Kandori, M., Repeated games. In *The New Palgrave Dictionary of Economics*, 2008 (Palgrave Macmillan: New York).

Kirman, A., On mistake beliefs and resultant equilibria. In *Individual Forecasting and Aggregate Outcomes: Rational Expectations Examined*, edited by R. Frydman and E. Phelps, pp. 147–165, 1983 (Cambridge University Press: Cambridge).

Kirman, A., Whom or what does the representative individual represent? *J. Econ. Perspect.*, 1992, **6**, 117–136.

Kirman, A., Ants, rationality, and recruitment. *Q. J. Econ.*, 1993, **108**, 137–156.

Kirman, A., Reflections on interaction and markets. *Quant. Finance*, 2002, **2**, 322–326.

Lallouache, M., Jedidi, A. and Chakraborti, A., Wealth distribution: to be or not to be a Gamma? *Science and Culture (Kolkata, India)*, Special Issue on 'Econophysics', 2010, **76**, 478–484.

Lamper, D., Howison, S.D. and Johnson, N.F., Predictability of large future changes in a competitive evolving population. *Phys. Rev. Lett.*, 2002, **88**, 017902-1–017902-4.

LeBaron, B., Agent-based computational finance. In *Handbook of Computational Economics*, edited by L. Tesfatsion and K. Judd, Vol. 2, pp. 1187–1233, 2006a (Elsevier North Holland: Amsterdam).

LeBaron, B., Agent-based financial markets: matching stylized facts with style. In *Post Walrasian Macroeconomics*, edited by D.C. Colander, pp. 221–236, 2006b (Cambridge University Press: Cambridge).

Levy, M. and Solomon, S., Power laws are logarithmic Boltzmann laws. *Int. J. Mod. Phys. C*, 1996, **7**, 595–601.

Li, Y., Riolo, R. and Savit, R., Evolution in minority games. (I). Games with a fixed strategy space. *Physica A*, 2000a, **276**, 234–264.

Li, Y., Riolo, R. and Savit, R., Evolution in minority games. (II). Games with variable strategy spaces. *Physica A*, 2000b, **276**, 265–283.

Lux, T., Emergent statistical wealth distributions in simple monetary exchange models: a critical review. In *Proceedings of the Econophysics of Wealth Distributions*, edited by A. Chatterjee, S. Yarlagadda, and B.K. Chakrabarti, pp. 51–60, 2005 (Springer: Berlin).

Lux, T. and Marchesi, M., Volatility clustering in financial markets. *Int. J. Theor. Appl. Finance*, 2000, **3**, 675–702.

Lux, T. and Westerhoff, F., Economics crisis. *Nature Phys.*, 2009, **5**, 2–3.

Mandelbrot, B., The Pareto–Levy law and the distribution of income. *Int. Econ. Rev.*, 1960, **1**, 79–106.

Mandelbrot, B., The variation of certain speculative prices. *J. Business*, 1963, **36**, 394–419.

Maslov, S., Simple model of a limit order-driven market. *Physica A*, 2000, **278**, 571–578.

Matthes, D. and Toscani, G., On steady distributions of kinetic models of conservative economies. *J. Statist. Phys.*, 2007, **130**, 1087–1117.

Mike, S. and Farmer, J.D., An empirical behavioral model of liquidity and volatility. *J. Econ. Dynam. Control*, 2008, **32**, 200–234.

Mohanty, P.K., Generic features of the wealth distribution in ideal-gas-like markets. *Phys. Rev. E*, 2006, **74**, 011117-1–011117-4.

Muni Toke, I., 'Market making' in an order book model and its impact on the bid–ask spread. In *Econophysics of Order-Driven Markets*, edited by F. Abergel, B.K. Chakrabarti, A. Chakraborti and M. Mitra, pp. 49–64, 2010 (Springer: Milan).

Nirei, M. and Souma, W., A two factor model of income distribution dynamics. *Rev. Income Wealth*, 2007, **53**, 440–459.

O'Hara, M., *Market Microstructure Theory*, 1995 (Blackwell: Oxford).

Pareto, V., *Cours d'Economie Politique*, 1897 (Rouge: Lausanne).

Patriarca, M., Chakraborti, A. and Germano, G., Influence of saving propensity on the power law tail of wealth distribution. *Physica A*, 2006, **369**, 723–736.

Patriarca, M., Chakraborti, A. and Kaski, K., Statistical model with a standard gamma distribution. *Phys. Rev. E*, 2004a, **70**, 016104-1–016104-5.

Patriarca, M., Chakraborti, A. and Kaski, K., Gibbs versus non-Gibbs distributions in money dynamics. *Physica A*, 2004b, **340**, 334–339.

Patriarca, M., Chakraborti, A., Kaski, K. and Germano, G., Kinetic theory models for the distribution of wealth: Power law from overlap of exponentials. In *Proceedings of the Econophysics of Wealth Distributions*, edited by A. Chatterjee, S. Yarlagadda, and B.K. Chakrabarti, pp. 93–110, 2005 (Springer: Berlin).

Patriarca, M., Chakraborti, A., Heinsalu, E. and Germano, G., Relaxation in statistical many-agent economy models. *Eur. J. Phys. B*, 2007, **57**, 219–224.

Patriarca, M., Heinsalu, E. and Chakraborti, A., Basic kinetic wealth-exchange models: common features and open problems. *Eur. J. Phys. B*, 2010, **73**, 145–153.

Potters, M. and Bouchaud, J.P., More statistical properties of order books and price impact. *Physica A*, 2003, **324**, 133–140.

Preis, T., Golke, S., Paul, W. and Schneider, J.J., Statistical analysis of financial returns for a multiagent order book model of asset trading. *Phys. Rev. E*, 2007, **76**, 016108-1–016108-13.

Raberto, M., Cincotti, S., Focardi, S. and Marchesi, M., Agent-based simulation of a financial market. *Physica A*, 2001, **299**, 319–327.

Repetowicz, P., Hutzler, S. and Richmond, P., Dynamics of money and income distributions. *Physica A*, 2005, **356**, 641–654.

Sala-i Martin, X., The world distribution of income. NBER Working Paper Series, 2002.

Sala-i Martin, X. and Mohapatra, S., Poverty, inequality and the distribution of income in the G20. Columbia University, Department of Economics, Discussion Paper Series, 2002.

Samanidou, E., Zschischang, E., Stauffer, D. and Lux, T., Agent-based models of financial markets. *Rep. Prog. Phys.*, 2007, **70**, 409–450.

Satinover, J.B. and Sornette, D., Illusion of control in Time-Horizon Minority and Parrondo games. *Eur. Phys. J. B*, 2007, **60**, 369–384.

Savit, R., Manuca, R. and Riolo, R., Adaptive competition, market efficiency, and phase transitions. *Phys. Rev. Lett.*, 1999, **82**, 2203–2206.

Scafetta, N., Picozzi, S. and West, B.J., An out-of-equilibrium model of the distributions of wealth. *Quant. Finance*, 2004, **4**, 353–364.

Scalas, E., Garibaldi, U. and Donadio, S., Statistical equilibrium in simple exchange games I – Methods of solution and application to the Bennati–Dragulescu–Yakovenko (BDY) game. *Eur. Phys. J. B*, 2006, **53**, 267–272.

Scalas, E., Garibaldi, U. and Donadio, S., Erratum. Statistical equilibrium in simple exchange games I. *Eur. Phys. J. B*, 2007, **60**, 271–272.

Sen, A.K., *Collective Choice and Social Welfare*, 1971 (Oliver & Boyd: Edinburgh).

Shostak, F., The mystery of the money supply definition. *Q. J. Aust. Econ.*, 2000, **3**, 69–76.

Silva, A.C. and Yakovenko, V.M., Temporal evolution of the 'thermal' and 'superthermal' income classes in the USA during 1983–2001. *Europhys. Lett.*, 2005, **69**, 304–310.

Slanina, F., Inelastically scattering particles and wealth distribution in an open economy. *Phys. Rev. E*, 2004, **69**, 046102-1–046102-7.

Slanina, F., Critical comparison of several order-book models for stock-market fluctuations. *Eur. Phys. J. B*, 2008, **61**, 225–240.

Slanina, F., Mean-field approximation for a limit order driven market model. *Phys. Rev. E*, 2001, **64**, 056136-1–056136-5.

Smith, E., Farmer, J.D., Gillemot, L. and Krishnamurthy, S., Statistical theory of the continuous double auction. *Quant. Finance*, 2003, **3**, 481–514.

Solomon, S. and Levy, M., Spontaneous scaling emergence in generic stochastic systems. *Int. J. Mod. Phys. C*, 1996, **7**, 745–751.

Sornette, D., Multiplicative processes and power laws. *Phys. Rev. E*, 1998, **57**, 4811–4813.

Stigler, G., Public regulation of the securities markets. *J. Business*, 1964, **37**, 117–142.

Sysi-Aho, M., Chakraborti, A. and Kaski, K., Adaptation using hybridized genetic crossover strategies. *Physica A*, 2003a, **322**, 701–709.

Sysi-Aho, M., Chakraborti, A. and Kaski, K., Biology helps you to win a game. *Phys. Scripta T*, 2003b, **106**, 32–35.

Sysi-Aho, M., Chakraborti, A. and Kaski, K., Intelligent minority game with genetic crossover strategies. *Eur. Phys. J. B*, 2003c, **34**, 373–377.

Sysi-Aho, M., Chakraborti, A. and Kaski, K., Searching for good strategies in adaptive minority games. *Phys. Rev. E*, 2004, **69**, 036125-1–036125-7.

Thurner, S., Farmer, J.D. and Geanakoplos, J., Leverage causes fat tails and clustered volatility, Preprint, 2009.

Wyart, M. and Bouchaud, J.-P., Self-referential behaviour, overreaction and conventions in financial markets. *J. Econ. Behav. Organiz.*, 2007, **63**, 1–24.

Yakovenko, V.M. and Rosser, J.B., Colloquium: statistical mechanics of money, wealth, and income. *Rev. Mod. Phys.*, 2009, **81**, 1703–1725.