Available online at www.sciencedirect.com
SCIENCE DIRECT®
Physica A 369 (2006) 723–736
PHYSICA A
www.elsevier.com/locate/physa

Influence of saving propensity on the power-law tail of the wealth distribution

Marco Patriarca[a,1,*], Anirban Chakraborti[b], Guido Germano[a]

[a]*Fachbereich Chemie, Philipps-Universität Marburg, 35032 Marburg, Germany*
[b]*Department of Physics, Banaras Hindu University, Varanasi-221005, India*

Received 1 December 2005; received in revised form 26 December 2005
Available online 3 March 2006

Abstract

Some general features of statistical multi-agent economic models are reviewed, with particular attention to the dependence of the equilibrium wealth distribution on the agents' saving propensities. It is shown that in a finite system of agents with a continuous saving propensity distribution a power-law tail with Pareto exponent $\alpha = 1$ can appear also when agents do not have saving propensities distributed over the whole interval between zero and one. Rather, a power-law can be observed in a finite interval of wealth, whose lower and upper ends are shown to be determined by the lower and upper cutoffs, respectively, of the saving propensity distribution. It is pointed out that a cutoff of the power-law tail can arise also through a different mechanism, when the number of agents is small enough. Numerical simulations have been carried out by implementing a procedure for assigning saving propensities homogeneously, which results in a smoother wealth distributions and correspondingly wider power-law intervals than other procedures based on random algorithms.

Keywords: Economics; Econophysics; Wealth distribution; Power-laws; Statistical mechanics

1. Introduction

Statistical mechanical models of closed economy systems have received considerable attention in recent years due to the fact that they seem to predict realistic wealth and income distributions shapes from a simple underlying dynamics, similar to that of microscopic models of classical statistical mechanics [1–27]. For an overview of the current situation of these models see Ref. [27].

In fact, as found in empirical distributions, they can reproduce a Boltzmann distribution at intermediate and a power-law at higher values, see e.g. Refs. [28–32]. A power-law form in the tail of statistical distributions was observed more than a century ago by the economist Pareto [33], who found that the wealth of individuals

*Corresponding author. Tel.: +49 821 598 3229; fax: +49 821 598 3222.
E-mail addresses: marco.patriarca@mac.com (M. Patriarca), achakraborti@yahoo.com (A. Chakraborti), germano@staff.uni-marburg.de (G. Germano).
URL: http://homepage.mac.com/physicssquare/~marco, http://www.cmth.bnl.gov/~anirban/, http://www.staff.uni-marburg/~germano.
[1]Current address: Institut für Physik, Universität Augsburg, Universitätsstr. 1, 86135 Augsburg, Germany.

0378-4371/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2006.01.091

in a stable economy has a cumulative distribution $F(x) \propto x^{-\alpha}$, where α, the Pareto exponent, has a value between 1 and 2.

In this paper, we consider models in which N agents interact exchanging a quantity x, that can be interpreted as a measure of the agents' wealth, expressed in money units. Depending on the parameters of the model, in particular on the values of the saving propensities $\{\lambda_i\}$ $(i = 1, \dots, N)$ of the N agents, the equilibrium wealth distribution can be a simple Boltzmann distribution for $\lambda_i = 0$ [3,4,9], a Gamma distribution with a similar exponential tail but a well-defined mode $\bar{x} > 0$ for $\lambda_i = \lambda_0 > 0$ [1,2,6,10,14,15], or a distribution with a power-law tail for randomly distributed λ_i [16,18]. It has been recently recognized [18,23] that the observed power-law arises from the mixture of Gamma distributions corresponding to agents with different values of λ. That is, in systems where the saving propensity is distributed according to an arbitrary distribution function $g(\lambda)$, individual agents relax toward a Gamma distribution similarly to systems with a global saving propensity λ_0, with the important difference that in this case the various Gamma distributions corresponding to different λ's will mix in such a way to produce a power-law.

In order for the power-law $F(x) \propto x^{-1}$ to be produced, a special role is played by agents with values of the saving propensity close to $\lambda = 1$: altering this part of the λ-distribution can strongly modify the tail of the wealth distribution [18]. The aim of the present paper is to further investigate quantitatively this important point, by studying in general terms the relation between the saving propensity distribution and the wealth distribution tail. We begin in Section 2 by recalling the main features of statistical multi-agent models. In Section 3 we focus on the relation between saving propensity and wealth distribution. First, we show that when the number of agents is low, discreteness effects may show up as an upper cutoff of the wealth power-law tail. Then through numerical simulations we illustrate how—even when the number of agents is high—the lower and upper cutoff of the saving propensity distribution determine those of the wealth distribution power-law. This is further discussed with some examples of realistic wealth distributions. Results are summarized in Section 4.

2. Statistical multi-agent models

In statistical multi-agent models N agents interact with each other through pair interactions in which a quantity x, generally referred to as "wealth" in the following, is exchanged. Each agent i $(i = 1, \dots, N)$ is characterized by the wealth x_i and, possibly, by some parameters, such as the saving propensity λ_i. The time evolution of the system is carried out by extracting randomly at every time step two agents i and j, who exchange an amount of wealth Δx between them,

$$x_i' = x_i - \Delta x,$$
$$x_j' = x_j + \Delta x. \tag{1}$$

It can be noticed that in this way the quantity x is conserved during the single transactions, $x_i' + x_j' = x_i + x_j$, where x_i' and x_j' are the agent wealths after the transaction has taken place.

2.1. The basic model

In a basic version of the model Δx is assumed to have a constant value [3–5],

$$\Delta x = \Delta x_0, \tag{2}$$

or to be proportional to the initial wealths [1,2,9],

$$\Delta x = \bar{\varepsilon} x_i - \varepsilon x_j, \tag{3}$$

where ε is a random number uniformly distributed between zero and one and $\bar{\varepsilon} = 1 - \varepsilon$. The form of Δx defined by Eq. (3) produces a random reshuffling of the total wealth, given by the sum of the wealths of the two agents [9], since Eq. (1) can be rewritten as

$$x_i' = \varepsilon (x_i + x_j),$$
$$x_j' = \bar{\varepsilon} (x_i + x_j). \tag{4}$$

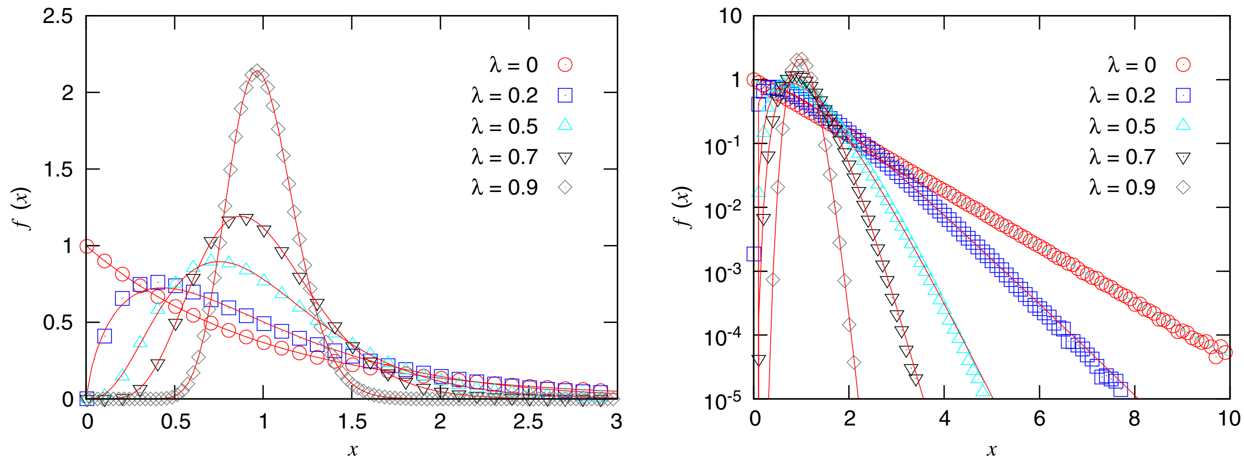

Fig. 1. Linear (left) and semi-log (right) plots of the probability density of wealth x from numerical simulations (dots) for various values of the global saving propensity λ in the interval [0, 1), compared with the theoretical curves (continuous lines) defined by Eqs. (8)–(10). The curve for $\lambda = 0$ is the Boltzmann distribution (5).

These dynamics rules, together with the constraint that transactions can take place only if $x_i' > 0$ and $x_j' > 0$, lead to an equilibrium state characterized by an exponential distribution,

$$f(x) = \langle x \rangle^{-1} \exp(-x/\langle x \rangle) \tag{5}$$

and one can identify the effective temperature T_λ of the system as the average wealth $\langle x \rangle$ (curve with $\lambda = 0$ in Fig. 1).

2.2. Models with a global saving propensity

A first generalization toward a more realistic model is based on the introduction of a saving criterion or, equivalently, of a dependence of the exchanged wealth Δx on the current wealths of the agents. In a model pioneered by Angle [1,2] and inspired by the Surplus Theory of economic development, one can have either a symmetrical or asymmetrical wealth exchange, depending on the relative richness of the two agents. In another model [10] agents save a fixed fraction λ (here referred to as the saving propensity, with $0 \le \lambda < 1$), independently of their current wealth. Both models lead to an equilibrium Gamma distribution. In the following we consider the latter model [10], whose evolution is described by the following equations:

$$x_i' = \lambda x_i + \varepsilon(1-\lambda)(x_i + x_j),$$

$$x_j' = \lambda x_j + \bar{\varepsilon}(1-\lambda)(x_i + x_j), \tag{6}$$

corresponding to a Δx in Eq. (1) given by

$$\Delta x = (1-\lambda)[\bar{\varepsilon} x_i - \varepsilon x_j]. \tag{7}$$

The corresponding equilibrium distribution is well fitted by the Gamma distribution [20,21]

$$f(\xi) = \frac{1}{\Gamma(D_\lambda/2)} \xi^{D_\lambda/2-1} \exp(-\xi) \equiv \gamma_{D_\lambda/2}(\xi) \tag{8}$$

as shown in Fig. 1. Here the dimensionless variable

$$\xi = \frac{x}{T_\lambda} \tag{9}$$

is just the variable x rescaled with respect to the effective temperature T_λ and

$$\frac{D_\lambda}{2} = 1 + \frac{3\lambda}{1-\lambda} = \frac{1+2\lambda}{1-\lambda},$$

$$T_\lambda = \frac{1-\lambda}{1+2\lambda}\langle x \rangle. \tag{10}$$

The parameter D_λ plays the role of an effective dimension, since the Gamma distribution $\gamma_n(\xi)$ given by Eq. (8) is identical to the Maxwell–Boltzmann distribution of kinetic energy for a system of molecules at temperature T_λ in D_λ dimensions (of course only for integer values of D_λ) [21,23]. In further support of this analogy, it is worth noting that T_λ and D_λ are related to each other through the equipartition theorem,

$$\langle x \rangle = \frac{D_\lambda T_\lambda}{2}. \tag{11}$$

The equivalence between kinetic theory and multi-agent economic models, suggested by the basic version of the kinetic multi-agent models [1,3–5,9], can thus be extended to values $\lambda > 0$ [23], as summarized in Table 1.

While λ varies from $\lambda = 0$ to 1, the effective dimension D_λ increases monotonically from 2 to ∞. In fact in a higher number of dimensions the fraction of kinetic energy exchanged between particles during a collision is smaller. At the same time, the market temperature T_λ decreases with increasing λ, showing smaller fluctuations of x during trades, consistently with a higher saving. One can notice that $T_\lambda = (1-\lambda)\langle x \rangle/(1+2\lambda) \approx (1-\lambda)\langle x \rangle$ is on average the amount of wealth exchanged during an interaction between agents, see Eqs. (6).

It is to be noticed that the saving propensity λ introduced in this and the following models does not necessarily represent the agent's investment strategy during a single wealth exchange. In fact stochastic processes provide in general a coarse grained description of the time evolution of a system: in the models considered here this means that one time step does not correspond to a single but rather to a large number of actual interactions between agents, the parameter λ only modeling the fraction of wealth which on average an agent saves in the end of them. These transactions may be for instance wealth exchanges of various type, trades, or investments, which are influenced by many other parameters of the global system. Thus the parameter λ is determined not only by the particular strategy of an agent, but also, e.g., on the ability of an agent to carry out a transaction in a fruitful way, on the specific possibilities of an agent to exploit a favorable situation or save resources, etc. The approximation which characterizes the type of models considered here is in the assumption that all these factors, when acting together and averaged over a large number of agent interactions, can be modeled by a single parameter λ. A natural extension of this model is then one with agents characterized by different saving propensities, as discussed in the following section.

Table 1
Analogy between physical kinetic and economic multi-agent model

	Physical model	Economic model
Exchanged quantity	E = energy	x = wealth
Units	N particles	N agents
Interaction	Collision	Trade or exchange
Dimension	Integer D	Real number D_λ
Equipartition theorem	$k_B T = 2\langle E \rangle / D$	$T_\lambda = 2\langle x \rangle / D_\lambda$
Reduced variable	$\eta = E/k_B T$	$\xi = x/T_\lambda$
Distribution	$f(\eta) = \gamma_{D/2}(\eta)$	$f(\xi) = \gamma_{D_\lambda/2}(\xi)$

2.3. Models with a continuous distribution of saving propensity

The basic model and the model with a global saving propensity produce equilibrium distribution functions—the Boltzmann and the Gamma distributions, respectively—that have been shown to interpolate well real data at small and intermediate values of wealth [1,2,6,28,34]. However, they still predict a distribution which is qualitatively different from the power-law found empirically at large values of wealth [27].

This is instead accomplished by various statistical models in which agents have realistically been diversified from each other, e.g. by assigning them different saving propensities λ_i [15,16,18,19,22,23,35,36]. In particular, uniformly distributed λ_i in the interval [0, 1), closed at $\lambda = 0$ and open at $\lambda = 1$, have been studied numerically in Refs. [16,18]. This model is described by the trading rule

$$x'_i = \lambda_i x_i + \varepsilon[(1-\lambda_i)x_i + (1-\lambda_j)x_j],$$

$$x'_j = \lambda_j x_j + \bar{\varepsilon}[(1-\lambda_i)x_i + (1-\lambda_j)x_j], \tag{12}$$

or, equivalently, by a Δx—as defined in Eq. (1)—given by

$$\Delta x = \bar{\varepsilon}(1-\lambda_i)x_i - \varepsilon(1-\lambda_j)x_j. \tag{13}$$

One of the main features of this model, which is supported by theoretical considerations [19,22,35–37], is that the wealth distribution exhibits a robust power-law tail,

$$f(x) \propto x^{-\alpha-1}, \tag{14}$$

with the Pareto exponent $\alpha = 1$ largely independent of the details of the λ-distribution.

An example of such a distribution is shown in Fig. 2-top left for a system of 1000 agents with a saving propensity λ_i uniformly distributed in the interval $0 \leq \lambda < 1$.

3. Saving propensity and wealth distribution

In this section, we consider systems of interacting agents whose saving propensities are continuously distributed between 0 and 1 with a given probability density $g(\lambda)$, in order to study how some features of $g(\lambda)$ affect the equilibrium form of the wealth distribution $f(x)$.

3.1. Wealth distribution at small and intermediate values of wealth

Even if the equilibrium distribution obtained from the basic model, that is the simple exponential function in Eq. (5), well describes real distributions at intermediate values of x, it predicts neither a significant fraction of rich agents nor a power-law tail. The fraction of agents outside a given interval $(0, x)$—that is the complementary cumulative distribution function—has a pure exponential form, $F(x) = \exp(-x/\langle x \rangle)$ and the mode of the distribution is $\bar{x} = 0$, so that most of the agents have a wealth close to zero. Real data suggest a mode $\bar{x} > 0$, which can be obtained through the introduction of a global saving propensity $\lambda > 0$ [1,2,6,10] since it leads to an equilibrium Gamma distribution [1,2,6,20,21], see Fig. 1.

3.2. Power-law distribution as a mixture of Gamma distributions

A remarkable feature of the equilibrium distribution, noticed in Ref. [18], is that—in spite of the resulting power-law tail—the individual partial wealth distributions of single agents with a given λ_i are not of a power-law type, but have a well defined mode and an exponential tail, similarly to the case of a global saving propensity. In fact we found [23] that the overall distribution can be resolved as a mixture of individual Gamma distributions with λ-dependent parameters. In particular, the mode and the average value of the partial distributions diverge as $\langle x(\lambda) \rangle \sim 1/(1-\lambda)$ for $\lambda \to 1$ [18,23].

Fig. 2 refers to a system of N agents with random saving propensities uniformly distributed between $\lambda = 0$ and 1 and illustrates how agents with different values of λ contribute to the power-law tail. As shown in Fig. 2-top right, when the power-law is resolved into the contributions from agents with saving propensity in

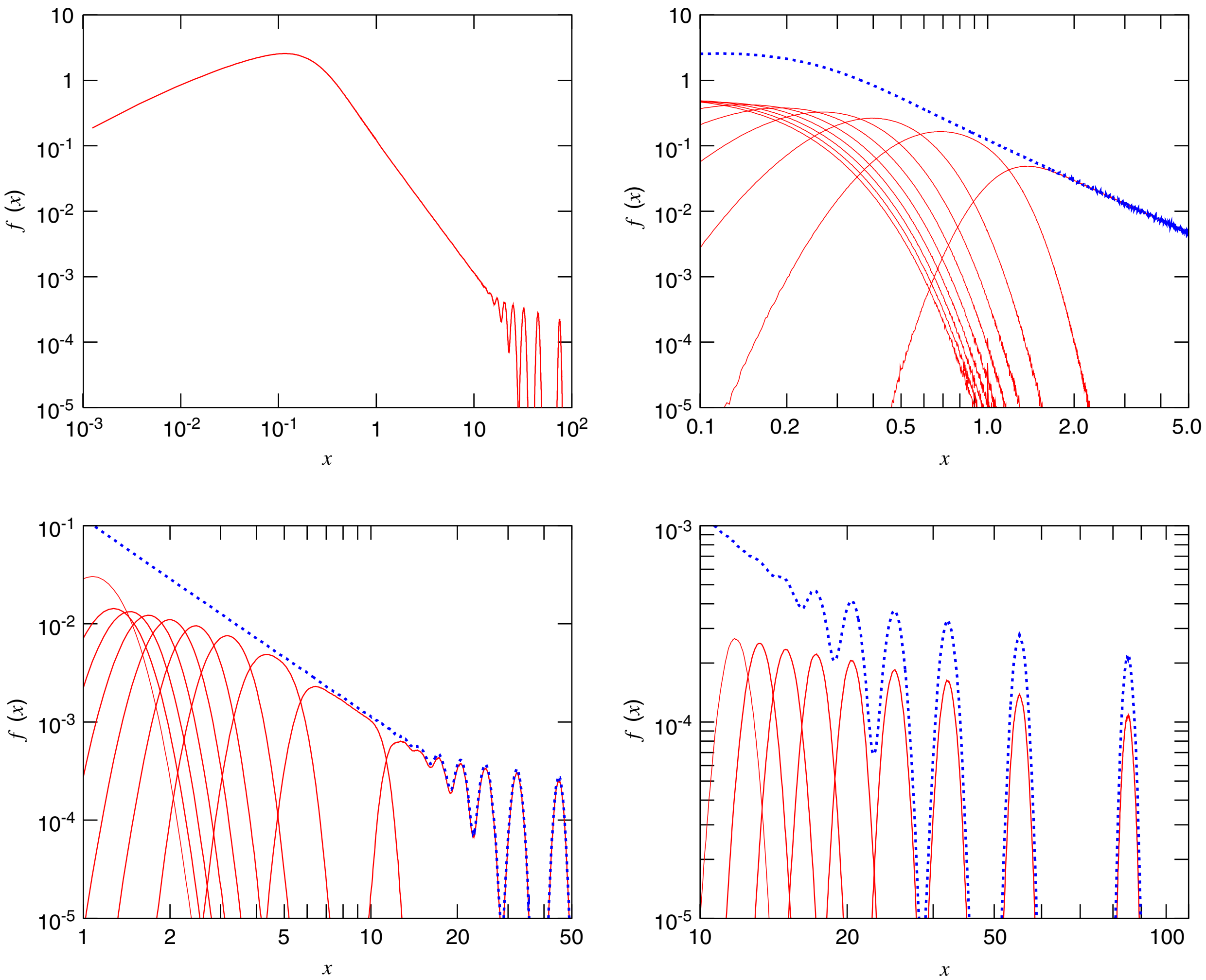

Fig. 2. Wealth distribution in a system of 1000 agents with saving propensities uniformly distributed in the interval $0 \leq \lambda < 1$. Top left: total distribution on the whole x-range. Top right: total (dotted line) and partial (continuous lines) distributions. Each continuous line (from left to right) represents the partial distribution of agents with $\lambda \in [j\Delta\lambda, (j+1)\Delta\lambda)$, with $\Delta\lambda = 0.1$ and $j = 0, \ldots, 9$. Bottom-left: the partial distribution of agents with $\lambda \in [0.9, 1)$ is further resolved into the contributions from the subintervals $\lambda \in [0.9 + j\Delta\lambda, 0.9 + (j+1)\Delta\lambda)$ with $\Delta\lambda = 0.01$. Bottom-right: the partial distribution of agents with $\lambda \in [0.99, 1)$ is further resolved into those from the subintervals $\lambda \in [0.99 + j\Delta\lambda, 0.99 + (j+1)\Delta\lambda)$, with $\Delta\lambda = 0.001$.

the subintervals $\lambda = [0, 0.1), [0.1, 0.2)), \ldots, [0.9, 1)$, one finds Gamma distributions for all the intervals—but the last one, $\lambda \in [0.9, 1)$ with the highest λ-values, which instead presents a power-law tail.

This result suggests the following remarks. First, it shows that if the upper limit of the λ-interval considered is held fixed at $\lambda = 1$, while the lower limit is varied—e.g. from $\lambda = 0$ for the total λ-interval to $\lambda = 0.9$ for the interval $\lambda = [0.9, 1)$—only the lower end of the distribution, where the power-law tail begins, is modified. Secondly, agents with $\lambda < 0.9$ do not seem to contribute to the asymptotic power-law form of the equilibrium distribution, which is instead due to agents from the sub-interval $\lambda \in [0.9, 1)$. This confirms the importance of agents with λ close to 1 for the power-law probability distribution $f(x) \propto x^{-\alpha-1} = x^{-2}$ [18]. Furthermore, this implies that the introduction of a finite cutoff λ_M should drastically influence the power-law tail.

In order to confirm the latter point, we have studied how the power-law distribution depends on selected agents with values of λ even closer to 1. However, reiterating the analysis on the partial distribution corresponding to the last subinterval $\lambda = [0.9, 1)$ led to the results shown in Fig. 2-bottom left: a power-law shape is observed only in a small region for agents with $\lambda > 0.98$, while at $x \approx 20$ the power-law breaks down.

Repeating the analogous analysis on the last subinterval $\lambda \in [0.99, 1)$ produces partial distributions different from a power-law, which can be resolved into almost disjoint contributions (Fig. 2-bottom right). Such a situation prevents any further investigation of the high-λ region in the system under consideration.

There are at least two different reasons why the breakdown of the power-law can happen, considered in greater detail in the following sections: The finite number of agents—i.e., the discreteness of the system—and the presence of a finite maximum value λ_M in the λ-distribution. These two factors are related—but not equivalent—to each other: while in a finite system there is necessarily a finite cutoff in the λ-distribution, equal to the maximum saving propensity $\lambda_M = \max\{\lambda_i\}$, the breakdown of the power-law can take place at $\lambda = \lambda^* < \lambda_M$ when the number of agents is small enough.

3.3. Effect of the discreteness of the system

A first reason for the breakdown of the power-law at large x, whose effects are visible in Fig. 2, is the discrete nature of the system, i.e., the finite number of agents N. In fact it has been shown that the peaks visible at large x after the power-law tail are due to single agents with high saving propensities [23,38].

This effect can be explained as due to the relative magnitude of the average wealth and the corresponding standard deviation of single-agent distributions for $\lambda \to 1$: given two agents with consecutive values of saving propensity λ_i and λ_{i+1}, the distance between the average values of the respective equilibrium Gamma distributions grows faster than the corresponding fluctuations [23]. It follows that in a system with a finite number of agents N, with saving propensities distributed between zero and one according to a continuous distribution $g(\lambda)$, there is a critical value $x^*(N)$ beyond which the wealth distribution is not a monotonous decreasing function of x and can be resolved into contributions from single agents. This fact may be important for the statistics of real systems: The minimum resolution Δx needed in order to obtain a statistically meaningful histogram will grow with x and in any case depend on the x-cutoff of the distribution. If in place of the probability distribution function one uses the cumulative distribution, the single agent contributions will show up as sharp steps.

However, as discussed below in Section 3.5, the situation is different when the λ-distribution has a cutoff $\lambda_M < 1$. In this case discreteness effects can be prevented from appearing, since the cutoff λ_M induces a corresponding finite cutoff x_M in the equilibrium x-distribution: one can safely study the limit $\lambda \to 1$ if one chooses N large enough that $x_M < x^*$.

3.4. Comparison of different ways to assign saving propensities

Before considering λ-distributions with a cutoff $\lambda_M < 1$, there is still much one can do, even in systems with λ distributed in the whole interval $[0, 1)$, in order to improve the smoothness of the x-distribution at large x and push the critical value x^*, at which the power-law breakdown takes place, significantly further. This is relevant especially when performing numerical simulations.

The peak structure observed at $x > x^*$ strongly depends on the particular λ-configuration assigned at the beginning of the simulation. This follows from the fact that the distance $\lambda_{i+1} - \lambda_i$ between two consecutive saving propensity values—the $\{\lambda_i\}$'s are assumed to be labeled in increasing order—is more amplified the closer the values of λ_i and λ_{i+1} are to $\lambda = 1$. In fact it grows as $1/(1-\lambda)^2$ [23]. Even if the average value of $\lambda_{i+1} - \lambda_i$ is $1/N$, larger distances will exist—due to random fluctuations—and will be mostly responsible for the resolvability of the distribution into contributions from single agents. In other words, x^* is very sensitive to the fluctuations of the (randomly assigned) distance $\lambda_{i+1} - \lambda_i$. Such fluctuations can be minimized by making the λ-configuration as homogeneous as possible. This can be achieved through a *homogeneous assignment* of the λ values, which can be realized as explained in Appendix A. As a simple example, here we consider a uniform λ-distribution in the interval $[0, 1)$. Following the usual procedure, N extractions from a uniform generator of random numbers in $(0, 1)$ are used to set the values $\{\lambda_i\}$ $(i = 1, \ldots, N)$. Alternatively, one can assign analytically $\lambda_i = i/N$ $(i = 0, \ldots, N-1)$. In this case one still recovers the same uniform distribution $g(\lambda) = 1$ for $0 \leq \lambda < 1$ in the continuous limit $(N \to \infty)$, but from a distribution which is originally already smooth, in that the distances $\lambda_{i+1} - \lambda_i = 1/N$ are constant.

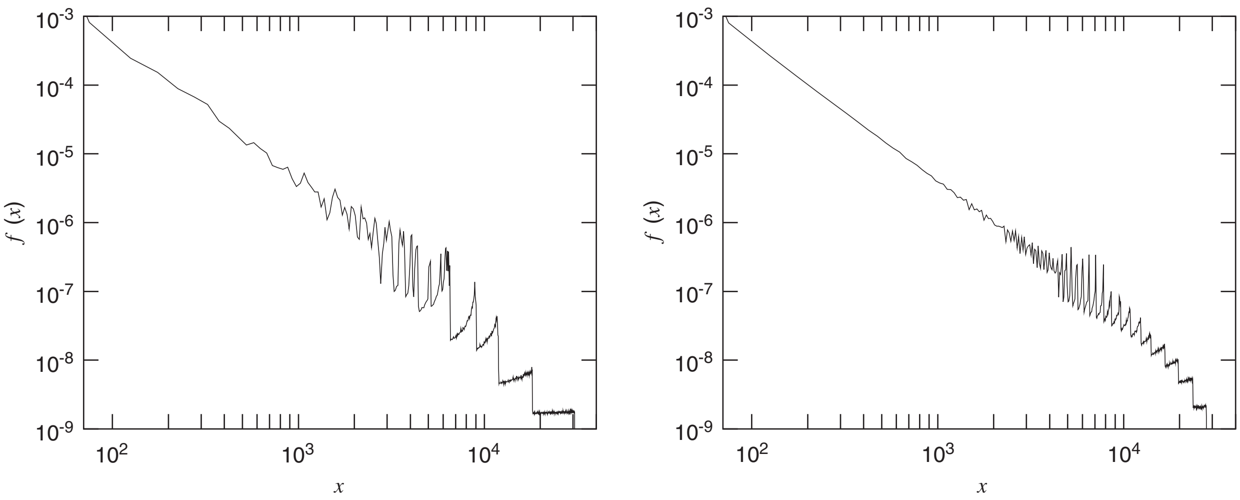

Fig. 3. Equilibrium wealth distributions in a system with $N = 10^6$ agents after 10^{12} trades. Left: wealth distribution from a λ-distribution obtained by random generation of the λ-values in the interval $[0, 1)$. Right: from a uniform λ-distribution obtained by setting $\lambda_i = i/N$, with $i = 0, \ldots, N-1$. Both the λ-distributions used reduce, in the continuous limit, to a uniform λ-distributions in the interval $[0, 1)$. The differences observed between the corresponding distributions are to be related to the method used to define the λ_i.

The differences between the two ways of assigning the λ-distribution are best shown through the comparison of the corresponding equilibrium wealth distributions: in the example of Fig. 3 the x-distribution on the left has been obtained by a random extraction of the values of λ_i, while the distribution on the right has been obtained by setting $\lambda_i = i/N$ $(i = 0, \ldots, N-1)$. In the rest of the paper we use the homogeneous—rather than the random—assignment of the λ_i for its convenience in numerical simulations, since it leads to a larger x^* and therefore a wider x-interval in which the power-law is observed.

3.5. *Influence of a saving propensity cutoff* λ_{M}

In this section, we study systems of interacting agents with saving propensities distributed uniformly on a subinterval of the λ-range. We assume a λ-distribution with an upper cutoff λ_{M},

$$g(\lambda) = \begin{cases} \lambda_{\mathrm{M}}^{-1} & \text{for } 0 \le \lambda \le \lambda_{\mathrm{M}}, \\ 0 & \text{for } \lambda_{\mathrm{M}} < \lambda < 1. \end{cases} \tag{15}$$

A system of $N = 10^5$ agents, with saving propensities distributed according to Eq. (15), has been studied by varying the cutoff λ_{M} between 0.9 and 0.9999. The chosen number of agents N is sufficient to avoid the appearance of the discreteness effects for all the λ-distributions considered.

The most interesting change in the equilibrium wealth distribution when λ_{M} is varied concerns the large x-region. A power-law with the same Pareto exponent $\alpha = 1$ is observed in a finite x-interval, which shrinks when λ_{M} is decreased, eventually disappearing at $\lambda_{\mathrm{M}} \approx 0.9$, and enlarges for growing λ_{M}, the only upper limit to it being represented either by the discreteness effects discussed above or the x-range considered. Results are shown in Fig. 4, in which curves from left to right correspond to increasing values of cutoff from $\lambda_{\mathrm{M}} = 0.9$ to $= 0.9997$.

These results show that a continuous λ-distribution extending as far as a finite cutoff $\lambda = \lambda_{\mathrm{M}} < 1$ is sufficient to have a power-law tail in the x-distribution, with the additional condition that the number of agents N is high enough to prevent a breakdown of the power-law due to discreteness effects.

Even if contradictory at first sight, it is still true that agents with λ close to one are those who determine the power-law tail, as suggested by numerical experiment [18]: the transition from an exponential to a power-law form of the distribution tail does not proceed through a global change of the functional form of the

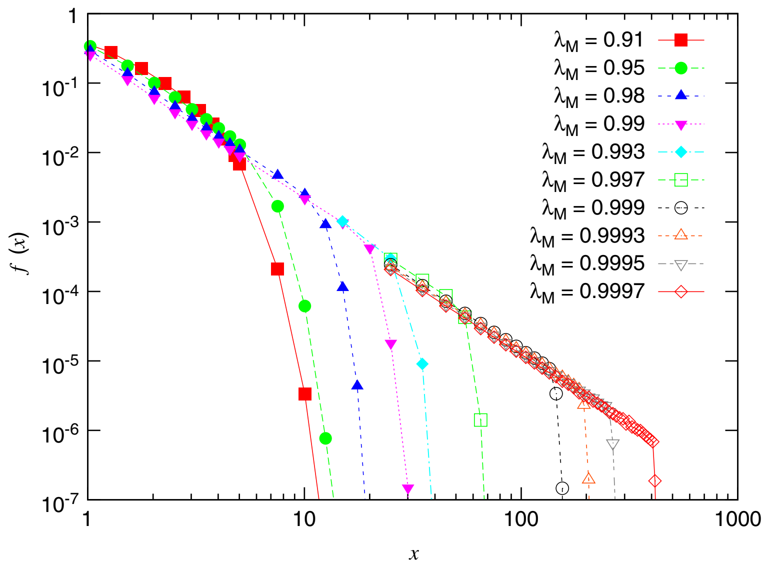

Fig. 4. Wealth distribution obtained for the uniform saving propensity distributions $g(\lambda)$ defined by Eq. (15). There are 10^5 agents in the system, with uniformly distributed saving propensities determined as explained in Appendix A. Curves from left to right correspond to increasing values of the cutoff λ_M between 0.9 and 0.9997.

distribution, but takes place gradually and continuously, as the cutoff λ_M is increased beyond a critical value $\lambda_M \approx 0.9$ toward $\lambda_M = 1$, through an enlargement of the x-interval in which the power-law is observed. By including additional agents with values of λ closer and closer to one, also the cutoff x_M increases and eventually, for $\lambda_M \to 1$, the power-law will extend to the whole positive x-axis. The latter statement cannot be proved by numerical simulation, but finds its justification in the match between the conclusions suggested by the numerical results obtained for various types of λ-distributions [18] and cutoff values on one hand and the theoretical predictions for $\lambda_M \equiv 1$ on the other hand [19,22,35,36].

3.6. Constructing a more realistic distribution

In real wealth distributions, an exponential form at intermediate values of wealth is known to coexist with a power-law tail at larger values. The power-law is due to a small percentage of population, of the order of a few percent, while the majority of the population with smaller average wealth contribute to the exponential part.

One can construct a more realistic example of wealth distribution than the exponential or power-law form alone, starting from the following information:

- A global saving propensity $\lambda_0 > 0$ is associated to an equilibrium Gamma distribution with a mode $\bar{x} > 0$ and an exponential tail.
- Agents with high λ's ($\lambda \approx 0.9$ to 1) produce a power-law tail, even when there is a finite cutoff in the λ-distribution.

These considerations when taken together suggest that one can construct a more realistic wealth distribution by choosing a suitable hybrid λ-distribution, similarly to what has been done in Ref. [16]: a small fraction of agents p_0 with saving propensities λ_i uniformly distributed in the interval $[0, 1)$ according to Eq. (A.4) and the remaining fraction $1 - p_0$ with a constant value of the saving propensity λ_0. The distribution corresponding to $p_0 = 0.01$ and $\lambda_0 = 0$ is shown in Fig. 5. Both an exponential shape at small x-scale and a power-law with exponent $\alpha = -1$ at large x are observed. It is noteworthy that the condition for the coexistence of an exponential and power-law form sets p_0 to a few percents, in agreement with real data on wealth distributions [27]. In fact, for larger values of p_0 the exponential part shrinks and the power-law dominates the distribution. It is also to be noticed that, due

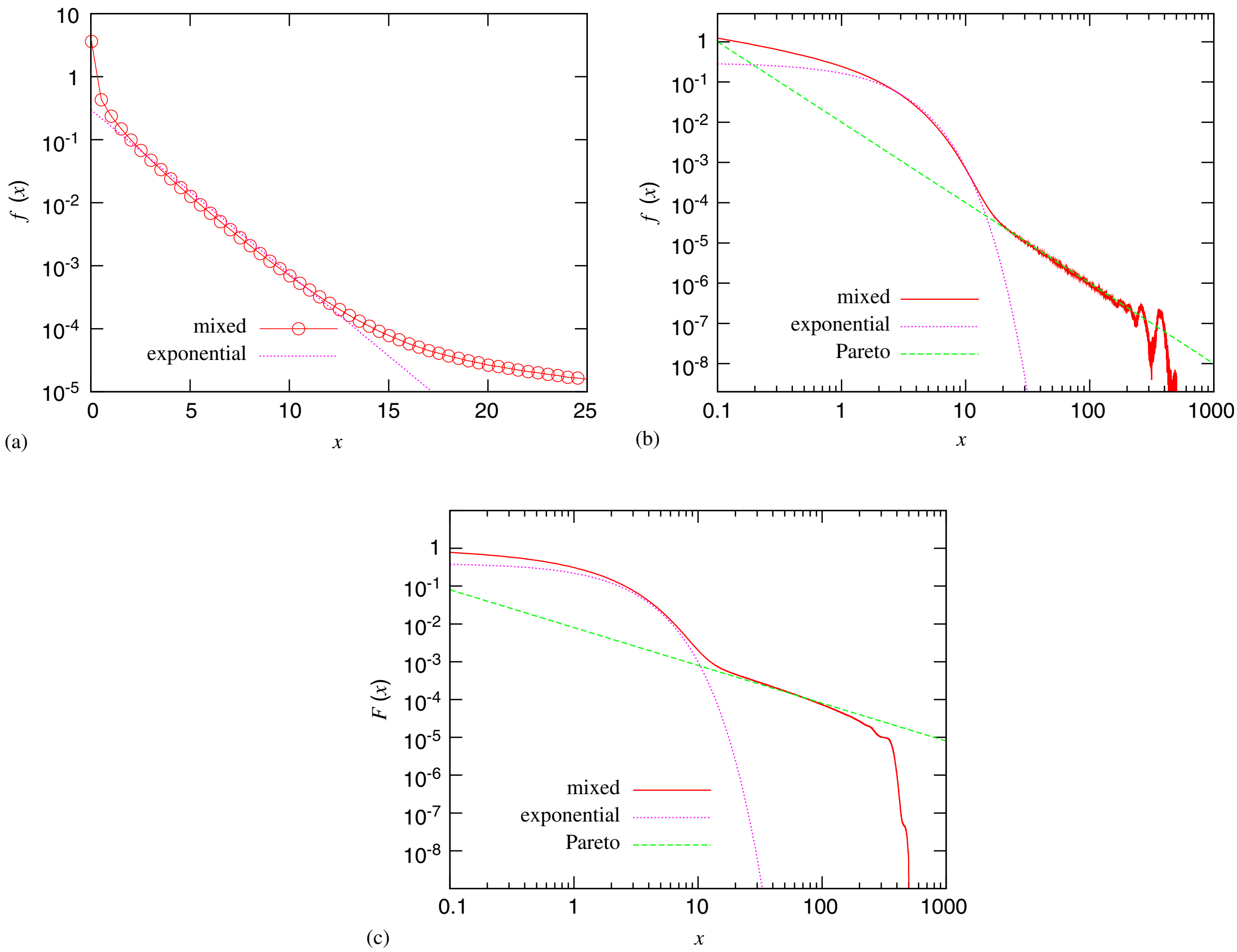

Fig. 5. Equilibrium wealth distribution of a population of 10^5 agents, in which one percent (10^3 agents) have uniformly distributed saving propensities in $\lambda = [0, 1)$, while the rest have $\lambda = 0$. (a) (semi-log scale): the numerical distribution $f(x)$ (circles) is compared in the small x-region with an exponential function $\propto \exp(-3x/5)$ (dotted line). (b) (log scale): The numerical distribution $f(x)$ (continuous line) is compared with a power-law function $\propto x^{-2}$ (dashed line) and the same exponential function of (a) (dotted line). The peaks visible at high x are due to the discreteness effects discussed in the text. (c) (log scale): The complementary cumulative distribution function $F(x) = \int_x^\infty \mathrm{d}y f(y)$ (continuous line) is compared with the power-law x^{-1} (dashed line) and an analogous exponential function $\propto \exp(-3x/5)$ (dotted line).

to the choice $\lambda_0 = 0$, the distribution in Fig. 5 has a mode $\bar{x} = 0$. By choosing a more realistic value $\lambda_0 = 0.2$ for $(1 - p_0) = 99\%$ of the agents and a uniform λ-distribution for the remaining $p_0 = 1\%$, one still obtains a distribution in which an exponential and a power-law tail coexist, but also with a mode $\bar{x} > 0$, see Fig. 6. The observed x-cutoff is determined by the λ-cutoff of the saving propensity distribution.

The point we would like to address by this example is that a realistic wealth distribution can be generated by a suitable tuning of the distribution parameters: the large fraction of agents with a small saving propensity $\lambda_0 > 0$, producing a mode $\bar{x} > 0$ and an exponential shape at intermediates values of x; the small fraction of agents with λ distributed over the whole interval $[0, 1)$, leading to a power-law tail; the cutoff $\lambda_M < 1$ of the saving propensity distribution inducing the corresponding finite cutoff x_M of the wealth distribution.

4. Conclusions

Within the framework of statistical multi-agent economic models with a continuous distribution of saving propensity, we have studied how a change in the shape of the saving propensity distribution influences the

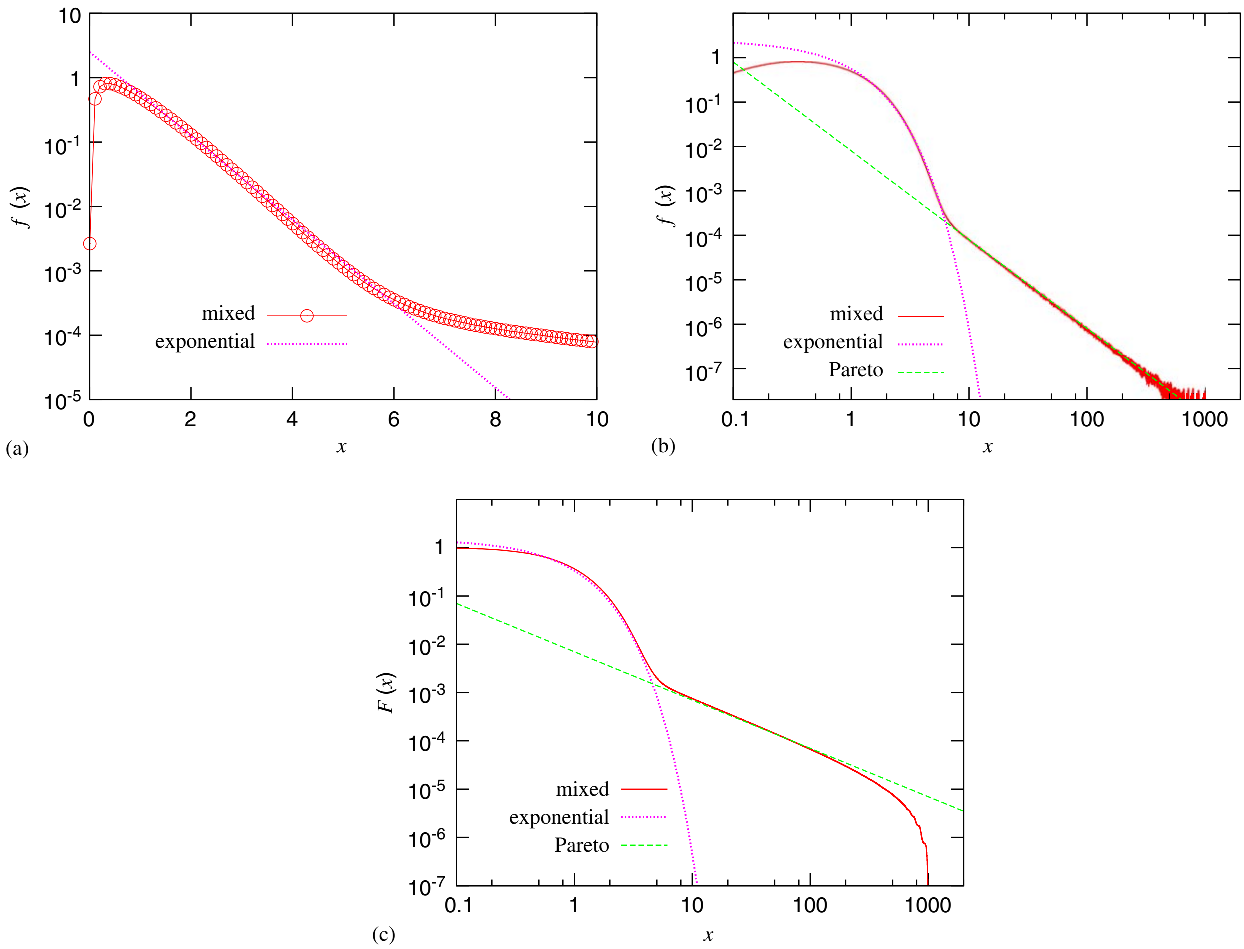

Fig. 6. Equilibrium wealth distribution of a population of 10^6 agents, in which 1% (10^4 agents) have uniformly distributed saving propensities $\lambda \in [0, 1)$, while the rest have $\lambda = 0.2$. With respect to the distribution in Fig. 5, this distribution has a mode $\bar{x} > 0$. The meaning of symbols is the same as in Fig. 5.

corresponding wealth distribution at equilibrium. In particular, we have shown that a continuous saving propensity distribution with $\lambda \in [0, \lambda_M]$ and $\lambda_M < 1$ generates a power-law at large values of wealth x on a *finite* x-interval extending as far as a certain $x = x_M$.

In general, the size of the power-law interval depends on the support of the λ-distribution: the upper (lower) end of (the continuous part of) the saving propensity distribution determines the upper (lower) end of the power-law in the corresponding equilibrium wealth distribution. We have also pointed out that a different mechanism may lead to the power-law breakdown, when the number of agents in the system is low enough. All this can provide useful criteria for modeling real systems and test whether the distributions of λ-distributions match according to this model.

Appendix A. Homogeneous assignment versus random generation of a finite ensemble with a given cumulative distribution

Here we consider two methods to generate a sequence of N numbers λ_i, with $i = 0, \ldots, N-1$. The $\{\lambda_i\}$ are assumed in the following examples to be defined in the interval $\lambda \in [0, 1)$ and become distributed according to a given distribution function $g(\lambda) = \mathrm{d}G(\lambda)/\mathrm{d}\lambda$, where $G(\lambda)$ is the cumulative distribution function, in the continuous limit ($N \to \infty$). The first method is based on random extractions, while the second one on homogeneous assignments of the λ-values. While both methods are based on the inversion of the cumulative distribution function $G(\lambda)$, with $G(0) = 0$ and $G(1) = 1$, and become equivalent to each other in the continuous limit, they provide significantly different distributions for finite N.

A.1. Random extraction

A well known and simple method for the random extraction of a variable λ with probability distribution $g(\lambda) = \mathrm{d}G(\lambda)/\mathrm{d}\lambda$ employs a generator of uniform random numbers and is here recalled for completeness.

Were the variable λ uniformly distributed in $(0, 1)$, the probability to extract a value in the subinterval $(\lambda, \lambda + \mathrm{d}\lambda)$ would simply be $\mathrm{d}G = \mathrm{d}\lambda$, i.e., it would be given by the interval length itself.

For a variable λ with a generic probability distribution $g(\lambda) = \mathrm{d}G(\lambda)/\mathrm{d}\lambda \neq 1$, the corresponding probability is

$$\mathrm{d}G = g(\lambda)\mathrm{d}\lambda, \tag{A.1}$$

i.e., the interval $\mathrm{d}\lambda$ is now weighted by the probability density $g(\lambda)$. One can notice however on the left-hand side that the probability to obtain a value of the cumulative distribution between G and $G + \mathrm{d}G$, corresponding to the probability to find the independent variable between λ and $\lambda + \mathrm{d}\lambda$, is equal to the interval itself $\mathrm{d}G$, just as in the case of a uniformly distributed variable. That is, the cumulative distribution function G is a uniform random variable in $(0, 1)$. Thus, extracting G uniformly in $(0, 1)$ and inverting $G = G(\lambda)$, one automatically obtains a random variable λ distributed according to the distribution function $g(\lambda)$.

A.2. Deterministic assignment

A distribution, which becomes equivalent to the randomly extracted distribution in the limit $N \to \infty$, is based on a homogeneous assignment of the values λ_i and does not make use of random number generators. If the sequence $\{\lambda_i\}$ is labeled in order of increasing value, i.e., $0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_{N-1} < 1$, then

$$\Lambda(i) = \lambda_i \tag{A.2}$$

is a function increasing monotonously with i, which can then be inverted to express i as a function of λ_i and define the function

$$G(\lambda_i) = \frac{i}{N}, \quad i = 0, \ldots, N - 1. \tag{A.3}$$

This function represents the fraction of agents with saving propensity less than or equal to λ_i, so that it represents by definition the cumulative distribution function. In fact, in the continuous limit, $0 < G(\lambda) < 1$ for every λ, $G(\lambda \to 0) \to 0$, and $G(\lambda \to 1) \to 1$. A sequence $\{\lambda_i\}$ with a cumulative distribution function $G(\lambda)$ can be obtained by inverting Eq. (A.3) for $i = 0, \ldots, N - 1$.

This procedure can also be understood in the following way. In this problem the order of the assignment of the λ-values is not relevant, since they are static parameters characterizing the agents. Thus instead of assigning the λ_i's with the help of a random variable, one can take the discrete values $\lambda = \{0, 1/N, 2/N, \ldots, (N - 1)/N\}$ to obtain the most homogeneous λ-distribution available. One could make the distribution more homogeneous in $(0, 1)$ by shifting all the values $\{\lambda_i\}$ by $\Delta\lambda/2 = 1/2N$ but this correction will be neglected here.

Here are a few examples to illustrate the application of this method.

Uniform distribution: The cumulative distribution function of a variable λ defined on and uniformly distributed in $[0, 1)$ is $G(\lambda) = \lambda$. Then Eq. (A.3) directly provides the values of λ_i as

$$\lambda_i = \frac{i}{N}, \quad i = 0, \ldots, N - 1. \tag{A.4}$$

Uniform distribution with an upper cutoff $\lambda_{\mathrm{M}} < 1$: In this case the cumulative distribution is

$$G(\lambda) = \begin{cases} \lambda/\lambda_{\mathrm{M}} & \text{if } 0 \leq \lambda < \lambda_{\mathrm{M}}, \\ 1 & \text{if } \lambda_{\mathrm{M}} \leq \lambda < 1. \end{cases} \tag{A.5}$$

The values of λ_i are obtained by inverting Eq. (A.3) for $0 \le \lambda < \lambda_M$,

$$\lambda_i = \frac{i}{N}\lambda_M, \quad i = 0, \ldots, N-1. \tag{A.6}$$

Uniform distribution with a lower and an upper cutoff $\lambda_m < \lambda_M < 1$: Here

$$G(\lambda) = \begin{cases} 0 & \text{if } \lambda \in [0, \lambda_m), \\ (\lambda - \lambda_m)/(\lambda_M - \lambda_m) & \text{if } \lambda \in [\lambda_m, \lambda_M), \\ 1 & \text{if } \lambda \in [\lambda_M, 1). \end{cases} \tag{A.7}$$

Then with the help of Eq. (A.3) in $\lambda \in [\lambda_m, \lambda_M)$ one obtains

$$\lambda_i = \lambda_m + \frac{i}{N}(\lambda_M - \lambda_M), \quad i = 0, \ldots, N-1. \tag{A.8}$$

References

[1] J. Angle, The surplus theory of social stratification and the size distribution of personal wealth, in: Proceedings of the Social Statistics Section of the American Statistical Association, 1983, p. 395.
[2] J. Angle, The surplus theory of social stratification and the size distribution of personal wealth, Soc. Forces 65 (1986) 293.
[3] E. Bennati, La simulazione statistica nell'analisi della distribuzione del reddito: modelli realistici e metodo di Montecarlo, ETS Editrice, Pisa, 1988.
[4] E. Bennati, Un metodo di simulazione statistica nell'analisi della distribuzione del reddito, Riv. Int. Sci. Econ. Commer. (1988) 735.
[5] E. Bennati, Il metodo di Montecarlo nell'analisi economica, Rass. Lav. dellI'SCO 10 (1993) 31.
[6] J. Angle, Deriving the size distribution of personal wealth from "the rich get richer, the poor get poorer", J. Math. Soc. 18 (1993) 27.
[7] S. Ispolatov, P. Krapivsky, S. Redner, Wealth distributions in asset exchange models, Eur. Phys. J. B 2 (1998) 267.
[8] J.P. Bouchaud, M. Mezard, Wealth condensation in a simple model of economy, Physica A 282 (2000) 536.
[9] A. Dragulescu, V.M. Yakovenko, Statistical mechanics of money, Eur. Phys. J. B 17 (2000) 723.
[10] A. Chakraborti, B.K. Chakrabarti, Statistical mechanics of money: how saving propensity affects its distribution, Eur. Phys. J. B 17 (2000) 167.
[11] S. Solomon, Generalized Lotka Volterra (GLV) models of stock markets, in: G. Ballot, G. Weisbuch (Eds.), Applications of Simulation to Social Sciences, Hermes Science Publications, 2000, pp. 301–322 URL http://xxx.lanl.gov/abs/cond-mat/9901250.
[12] P. Richmond, S. Solomon, Power laws are disguised Boltzmann laws, Int. J. Mod. Phys. C 12 (2001) 333 URL http://xxx.lanl.gov/abs/cond-mat/0010222.
[13] S. Solomon, P. Richmond, Stable power laws in variable economies; Lotka-Volterra implies Pareto-Zipf, Eur. Phys. J. B 27 (2002) 257.
[14] A. Chakraborti, Distributions of money in model markets of economy, Int. J. Mod. Phys. C 13 (2002) 1315.
[15] J. Angle, The statistical signature of pervasive competition on wage and salary incomes, J. Math. Soc. 26 (2002) 217.
[16] A. Chatterjee, B.K. Chakrabarti, S.S. Manna, Money in gas-like markets: Gibbs and Pareto laws, Phys. Scr. T 106 (2003) 36.
[17] J.R. Iglesias, S. Goncalves, G. Abramson, J.L. Vega, Correlation between risk aversion and wealth distribution, Physica A 342 (2004) 186.
[18] A. Chatterjee, B.K. Chakrabarti, S.S. Manna, Pareto law in a kinetic model of market with random saving propensity, Physica A 335 (2004) 155.
[19] P. Repetowicz, S. Hutzler, P. Richmond, Dynamics of money and income distributions, Physica A 356 (2005) 641.
[20] M. Patriarca, A. Chakraborti, K. Kaski, Gibbs versus non-Gibbs distributions in money dynamics, Physica A 340 (2004) 334.
[21] M. Patriarca, A. Chakraborti, K. Kaski, Statistical model with a standard Γ distribution, Phys. Rev. E 70 (2004) 016104.
[22] A. Chatterjee, B.K. Chakrabarti, R.B. Stinchcombe, Master equation for a kinetic model of trading market and its analytic solution, Phys. Rev. E 72 (2005) 026126.
[23] M. Patriarca, A. Chakraborti, K. Kaski, G. Germano, Kinetic theory models for the distribution of wealth: power law from overlap of exponentials, in: A. Chatterjee, S. Yarlagadda, B.K. Chakrabarti (Eds.), Econophysics of Wealth Distributions, Springer, Berlin, 2005, p. 93.
[24] X. SalaiMartin, S. Mohapatra, Poverty, inequality and the distribution of income in the G20, Discussion Paper Series, Department of Economics, Columbia University, URL www.columbia.edu/cu/economics/discpapr/DP0203-10.pdf.
[25] X. Sala-i-Martin, The world distribution of income, NBER Working Paper Series, URL www.nber.org/papers/w8933.
[26] H. Aoyama, W. Souma, Y. Fujiwara, Growth and fluctuations of personal and company's income, Physica A 324 (2003) 352.
[27] A. Chatterjee, S.Yarlagadda, B.K. Chakrabarti (Eds.), Econophysics of Wealth Distributions, Springer, Berlin, 2005.
[28] A. Dragulescu, V.M. Yakovenko, Exponential and power-law probability distributions of wealth and income in the United Kingdom and the United States, Physica A 299 (2001) 213.
[29] A. Dragulescu, V.M. Yakovenko, Evidence for the exponential distribution of income in the USA, Eur. Phys. J. B 20 (2001) 585.

[30] Y. Fujiwara, W. Souma, H. Aoyama, T. Kaizoji, M. Aoki, Growth and fluctuations of personal income, Physica A 321 (2003) 598.
[31] M. Levy, S. Solomon, New evidence for the power-law distribution of wealth, Physica A 242 (1997) 90.
[32] S. Sinha, Evidence for power-law tail of the wealth distribution in India, Physica A 359 (2006) 555.
[33] V. Pareto, Cours d'economie politique, Rouge, Lausanne, 1897. (Reprinted as the first volume of Oeuvres Completes, Droz, Geneva, 1964.)
[34] J.C. Ferrero, The statistical distribution of money and the rate of money transference, Physica A 341 (2004) 575.
[35] A. Das, S. Yarlagadda, A distribution function analysis of wealth distribution, URL http://xxx.lanl.gov/abs/cond-mat/0310343.
[36] A. Das, S. Yarlagadda, An analytic treatment of the Gibbs–Pareto behavior in wealth distribution, Physica A 353 (2005) 529.
[37] P. Bhattacharyya, A. Chatterjee, B.K. Chakrabarti, A common origin of the power law distributions in models of market and earthquake, URL http://xxx.lanl.gov/abs/physics/0510038.
[38] K. Bhattacharya, G. Mukherjee, S.S. Manna, Detailed simulation results for some wealth distribution models in econophysics, in: A. Chatterjee, S. Yarlagadda, B.K. Chakrabarti (Eds.), Econophysics of Wealth Distributions, Springer, Berlin, 2005, p. 111.