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FAST TRACK COMMUNICATION

Quantum entanglement: the unitary 8-vertex braid matrix with imaginary rapidity

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Abstract
We study quantum entanglements induced on product states by the action of 8-vertex braid matrices, rendered unitary with purely imaginary spectral parameters (rapidity). The unitarity is displayed via the 'canonical factorization' of the coefficients of the projectors spanning the basis. This adds one more *new* facet to the famous and fascinating features of the 8-vertex model. The double periodicity and the analytic properties of the elliptic functions involved lead to a rich structure of the 3-tangle quantifying the entanglement. We thus explore the complex relationship between topological and quantum entanglement.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Are topological and quantum entanglements related? This intriguing question is recently being studied from different angles. One approach was initiated by Aravind [1]. Kauffman and Lomonaco [2] pointed out that braid matrices (representing the third Reidemeister move [3], fundamental in the topological study of knots and links) correspond to universal quantum gates, when they are *also* unitary. In joining the two ends of three separate strings to form three loops which have undergone two Reidemeister moves (type 3), one obtains a Borromean ring in which the strings are now topologically entangled and can no longer be separated (see [2] for a discussion and diagramatic illustration). Braid equations also embody matrix representations of the type-3 Reidemeister move. If unitary (but 'non-local') transformations induced on pure product states in a base space generate quantum entanglements of such states, one obtains an intriguing link between topological and quantum entanglements. In previous studies, unitary braid matrices were constructed explicitly for all dimensions [4] and applied to the study of quantum entanglements [5]. Here, our starting point is the remarkable 8-vertex model [6] with braid matrix related to the Yang–Baxter one through a suitable permutation of elements, rendered unitary by a passage to imaginary rapidity ($\theta \rightarrow \mathrm{i}\theta$). The consequent unitarity is displayed transparently through 'canonical factorization' [7] of the coefficients of the projectors. One now no longer has a statistical model with real, positive Boltzmann weights, but unitarity thus implemented opens a new road (as will be shown below) to quantum entanglements. We first formulate such unitarization ($\theta \rightarrow \mathrm{i}\theta$) in a general fashion and illustrate with the relatively simple 6-vertex case. Then we concentrate on the far more complex 8-vertex case, and study the 3-tangle [8] parametrized by sums of products of ratios of the q-Pochhammer functions.

2. Theory and results

2.1. Unitarity for imaginary rapidity

The $\widehat{R}(\theta)$ being an $N^2 \times N^2$ matrix acts on the base space $V_N \otimes V_N$ spanned by the tensor product of N-dimensional vectors V_N. Defining $\widehat{R}_{12}(\theta) = \widehat{R}(\theta) \otimes I_N$, $\widehat{R}_{23}(\theta) = I_N \otimes \widehat{R}(\theta)$, where I_N is the $N \times N$ identity matrix, the corresponding braid operator is

$$\begin{aligned} \widehat{B} &\equiv \widehat{R}_{12}(\theta)\widehat{R}_{23}(\theta + \theta')\widehat{R}_{12}(\theta') \\ &= \widehat{R}_{23}(\theta')\widehat{R}_{12}(\theta + \theta')\widehat{R}_{23}(\theta). \end{aligned} \tag{1}$$

The above Braid equation corresponds to the equivalence of knots related through the third Reidemeister move [3]. A useful introduction to the equivalent Yang–Baxter formalism is provided in [9]. Of course, $\widehat{B}$ acts on the base space $V_N \otimes V_N \otimes V_N$. Additionally, if the braid matrix $\widehat{R}$ is also *unitary*, then it induces unitary transformations in $V_N \otimes V_N$, and $\widehat{B}$ in $V_N \otimes V_N \otimes V_N$. It is crucial to note the essential point that a non-trivial unitary $\widehat{R}$ induces *non-local* unitary transformations. Had it been the case that $\widehat{R} = \widehat{R_1} \otimes \widehat{R_2}$, where $\widehat{R_1}$ is acting on V_1, $\widehat{R_2}$ on V_2 and $\widehat{R}$ on $V_1 \otimes V_2$, then such an $\widehat{R}$ would have been trivial from the point of braiding. Thus, a non-trivial $\widehat{B}$ induces a non-local transformation in $V_N \otimes V_N \otimes V_N$.

The non-local unitary actions set the stage for quantum entanglements. It was shown [5] that $\widehat{B}$, acting on *unentangled* product states of the general form

$$|i\rangle \otimes |j\rangle \otimes |k\rangle \equiv \left(\sum_{i=1}^{N} x_i |a_i\rangle\right) \otimes \left(\sum_{i=1}^{N} y_i |b_i\rangle\right) \otimes \left(\sum_{i=1}^{N} z_i |c_i\rangle\right)$$

in $V_N \otimes V_N \otimes V_N$, can generate entanglements for certain choices. We also studied entanglements generated by two different classes (real and complex) of $\widehat{B}$. The '3-tangles' and '2-tangles' characterizing such entanglements were obtained explicitly in parametrized forms in terms of the parameters of $\widehat{B}$, and the variations with (θ, θ') were analyzed.

In another paper [7], we introduced the 'canonical factorization' for $\widehat{R}(\theta)$, which turns out to be very significant. The whole development is not necessary here, and so we summarize below only the essential steps of that formalism. We show that the simple passage ($\theta \rightarrow \mathrm{i}\theta$) is sufficient to provide unitarity under the following constraints:

(i) $\widehat{R}(\theta) = \sum_i \dfrac{f_i(\theta)}{f_i(-\theta)} P_i$, where $P_i P_j = \delta_{ij} P_i$ and $\sum_i P_i = I_{N^2} (= I_N \otimes I_N)$;

(ii) $(\widehat{R}(\theta))_{\mathrm{trans}} = \widehat{R}(\theta)$.

Thus, initially $\widehat{R}(\theta)$ is *real* and *symmetric*, with a *complete* set of orthonormal projectors P_i as a basis. The domain of i depends on the class considered. The factorized form

$f_i(\theta)/f_i(-\theta)$ of the coefficients in the first constraint might seem strongly restrictive, but in fact it was shown that it holds true for all well-known standard cases, and new such cases were constructed [7], with the new term 'canonical factorization' being introduced. One can easily check that a direct consequence of the constraints is $\widehat{R}(\theta)\widehat{R}(-\theta) = I \otimes I$. After the passage $(\theta \to \mathrm{i}\theta)$, since P_j's are real, one can easily show that $(\widehat{R}(\mathrm{i}\theta))^\dagger(\widehat{R}(\mathrm{i}\theta)) = I \otimes I$, i.e. $\widehat{R}(\mathrm{i}\theta)$ is *unitary*.

2.2. 6-vertex model

First, we demonstrate this formalism with the simpler case of the 6-vertex models. Following [7], which contains an extensive classification of 'canonical factorization' for all dimensions, we define the projectors

$$P_{1(\pm)} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{vmatrix}, \qquad P_{2(\pm)} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm 1 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \tag{2}$$

and obtain for the *ferroelectric* case after $(\theta \to \mathrm{i}\theta)$, with the real parameter γ,

$$\widehat{R}(\mathrm{i}\theta) = P_{1(+)} + P_{1(-)} + \frac{\cosh\frac{1}{2}(\gamma - \mathrm{i}\theta)}{\cosh\frac{1}{2}(\gamma + \mathrm{i}\theta)} P_{2(+)} + \frac{\sinh\frac{1}{2}(\gamma - \mathrm{i}\theta)}{\sinh\frac{1}{2}(\gamma + \mathrm{i}\theta)} P_{2(-)}, \tag{3}$$

which evidently satisfies the unitarity constraint.

2.3. 8-vertex model

Now we proceed to the more complicated case of the 8-vertex model. We again define the projectors as in equation (2). The coefficients are expressed [10] in terms of infinite products (q-Pochhammer functions), starting with

$$(x; a)_\infty = \prod_{n\geqslant 0}(1 - xa^n). \tag{4}$$

Setting $z = \exp(\theta)$, the initial 8-vertex matrix is

$$\widehat{R}(\theta) = (a+d)P_{1(+)} + (a-d)P_{1(-)} + (c+b)P_{2(+)} + (c-b)P_{2(-)}, \tag{5}$$

where with supplementary real parameters p, q one obtains [7]

$$(a \pm d) = \frac{\left(\mp p^{\frac{1}{2}}q^{-1}z;\, p\right)_\infty \left(\mp p^{\frac{1}{2}}qz^{-1};\, p\right)_\infty}{\left(\mp p^{\frac{1}{2}}q^{-1}z^{-1};\, p\right)_\infty \left(\mp p^{\frac{1}{2}}qz;\, p\right)_\infty} \tag{6}$$

$$(c \pm b) = \frac{\left(q^{\frac{1}{2}}z^{-\frac{1}{2}} \pm q^{-\frac{1}{2}}z^{\frac{1}{2}}\right)}{\left(q^{\frac{1}{2}}z^{\frac{1}{2}} \pm q^{-\frac{1}{2}}z^{-\frac{1}{2}}\right)} \frac{(\mp pq^{-1}z;\, p)_\infty(\mp pqz^{-1};\, p)_\infty}{(\mp pq^{-1}z^{-1};\, p)_\infty(\mp pqz;\, p)_\infty}. \tag{7}$$

We note that defining the numerators of the two equations ((6) and (7)) as $f_{1(\pm)}(z)$ and $f_{2(\pm)}(z)$, respectively, and using the fact that $z = \exp(\theta)$, we can express them as $(a \pm d) = \frac{f_{1(\pm)}(z)}{f_{1(\pm)}(z^{-1})}$ and $(c \pm b) = \frac{f_{2(\pm)}(z)}{f_{2(\pm)}(z^{-1})}$, which implies that the essential property of the coefficients, 'canonical factorization', is preserved.

After $(\theta \to \mathrm{i}\theta)$ passage, we thus have $(a \pm d) = \frac{f_{1(\pm)}(\mathrm{e}^{\mathrm{i}\theta})}{f_{1(\pm)}(\mathrm{e}^{-\mathrm{i}\theta})}$, and $(c \pm b) = \frac{f_{2(\pm)}(\mathrm{e}^{\mathrm{i}\theta})}{f_{2(\pm)}(\mathrm{e}^{-\mathrm{i}\theta})}$. Since the other parameters are real, we can interpret the coefficients as new *phases* $(a \pm d) = \mathrm{e}^{\mathrm{i}\Psi_{(\pm)}}$ and $(c \pm b) = \mathrm{e}^{\mathrm{i}\Phi_{(\pm)}}$, where the *phase factors* $(\Psi_{(\pm)}, \Phi_{(\pm)})$ are complicated functions of (p, q, θ). Note also that the coefficients under complex conjugation become

$(a \pm d)^* = \frac{f_{1(\pm)}(\mathrm{e}^{-\mathrm{i}\theta})}{f_{1(\pm)}(\mathrm{e}^{\mathrm{i}\theta})} = (a \pm d)^{-1}$ and $(c \pm b)^* = \frac{f_{2(\pm)}(\mathrm{e}^{-\mathrm{i}\theta})}{f_{2(\pm)}(\mathrm{e}^{\mathrm{i}\theta})} = (c \pm b)^{-1}$. Since the projectors are real and symmetric, we again have the unitarity $(\widehat{R}(\mathrm{i}\theta))^\dagger \widehat{R}(\mathrm{i}\theta)) = I \otimes I$. This opens the door of a new domain as a generator of quantum entanglements, as shown hereafter.

2.3.1. Action of the braid operator on the base space. Consider the base space that is eight dimensional and spanned by the states $|\epsilon_1\rangle \otimes |\epsilon_2\rangle \otimes |\epsilon_3\rangle \equiv |\epsilon_1\epsilon_2\epsilon_3\rangle$, where $\epsilon_i = \pm, i = 1, 2, 3$. We will adopt a notation $(|+\rangle, |-\rangle) \to (|1\rangle, |\bar{1}\rangle)$ that generalizes smoothly to higher spins. The braid operator is

$$\widehat{B} = \widehat{B}^\dagger = (\widehat{R}(\mathrm{i}\theta) \otimes I_2)(I_2 \otimes \widehat{R}(\mathrm{i}\theta + \mathrm{i}\theta'))(\widehat{R}(\mathrm{i}\theta') \otimes I_2), \tag{8}$$

and the matrix

$$\widehat{R}(\mathrm{i}\theta) = \begin{vmatrix} a & 0 & 0 & d \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ d & 0 & 0 & a \end{vmatrix}, \tag{9}$$

where $(a \pm d) = \mathrm{e}^{\mathrm{i}\Psi_{(\pm)}(\theta)}$, $(c \pm b) = \mathrm{e}^{\mathrm{i}\Phi_{(\pm)}(\theta)}$ and

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\Psi_{(\pm)}(\theta)} &= \frac{\left(\mp p^{\frac{1}{2}} q^{-1} \mathrm{e}^{\mathrm{i}\theta}; p\right)_\infty \left(\mp p^{\frac{1}{2}} q \mathrm{e}^{-\mathrm{i}\theta}; p\right)_\infty}{\left(\mp p^{\frac{1}{2}} q^{-1} \mathrm{e}^{-\mathrm{i}\theta}; p\right)_\infty \left(\mp p^{\frac{1}{2}} q \mathrm{e}^{\mathrm{i}\theta}; p\right)_\infty} \\ \mathrm{e}^{\mathrm{i}\Phi_{(\pm)}(\theta)} &= \frac{q^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}} \pm q^{-\frac{1}{2}} \mathrm{e}^{\mathrm{i}\frac{\theta}{2}}}{q^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\frac{\theta}{2}} \pm q^{-\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}} \times \frac{\left(\mp p q^{-1} \mathrm{e}^{\mathrm{i}\theta}; p\right)_\infty \left(\mp p q \mathrm{e}^{-\mathrm{i}\theta}; p\right)_\infty}{\left(\mp p q^{-1} \mathrm{e}^{-\mathrm{i}\theta}; p\right)_\infty \left(\mp p q \mathrm{e}^{\mathrm{i}\theta}; p\right)_\infty}. \end{aligned} \tag{10}$$

One crucial fact is that $\widehat{R}$ has non-zero elements only on the diagonal and the anti-diagonal. This effectively splits the base space into two four-dimensional subspaces closed under the action of $\widehat{B}$. They are spanned respectively by $V_{(e)} \equiv (|111\rangle, |1\bar{1}\bar{1}\rangle, |\bar{1}1\bar{1}\rangle, |\bar{1}\bar{1}1\rangle)$ and $V_{(o)} \equiv (|\bar{1}\bar{1}\bar{1}\rangle, |\bar{1}11\rangle|1\bar{1}1\rangle, |11\bar{1}\rangle)$, corresponding to even and odd numbers of indices with bars. Moreover, for say

$$\widehat{B}|111\rangle = \alpha_1|111\rangle + \beta_1|1\bar{1}\bar{1}\rangle + \gamma_1|\bar{1}1\bar{1}\rangle + \delta_1|\bar{1}\bar{1}1\rangle, \tag{11}$$

one has

$$\widehat{B}|\bar{1}\bar{1}\bar{1}\rangle = \alpha_1|\bar{1}\bar{1}\bar{1}\rangle + \beta_1|\bar{1}11\rangle + \gamma_1|1\bar{1}1\rangle + \delta_1|11\bar{1}\rangle, \tag{12}$$

with the *same coefficients* $(\alpha_1, \beta_1, \gamma_1, \delta_1)$. More generally, the symmetry of (9) ensures for

$$\widehat{B}|ijk\rangle = c_1|ijk\rangle + c_2|i\bar{j}\bar{k}\rangle + c_3|\bar{i}j\bar{k}\rangle + c_4|\bar{i}\bar{j}k\rangle \tag{13}$$

with $i, j, k = (1 \text{ or } \bar{1})$, the direct consequence

$$\widehat{B}|\bar{i}\bar{j}\bar{k}\rangle = c_1|\bar{i}\bar{j}\bar{k}\rangle + c_2|\bar{i}jk\rangle + c_3|i\bar{j}k\rangle + c_4|ij\bar{k}\rangle. \tag{14}$$

The coefficients are conserved as above for $(i, j, k) \to (\bar{i}, \bar{j}, \bar{k})$. Thus, it is sufficient to evaluate the action of $\widehat{B}$ on the subspace $V_{(e)}$ or $V_{(o)}$.

2.3.2. Density matrices and 3-tangles. To study the behavior of density matrices and 3-tangles, we explicitly consider the action of $\widehat{B}$ on the product state $|1\rangle \otimes |1\rangle \otimes |1\rangle \equiv |111\rangle$, in the subspace $V_{(e)}$, given by (11). Some straightforward algebra gives

$$\begin{aligned} \alpha_1 &= f_{(+)} f'_{(+)} f''_{(+)} + f_{(-)} f'_{(-)} g''_{(+)} \\ \beta_1 &= g_{(+)} f'_{(+)} f''_{(-)} + g_{(-)} f'_{(-)} g''_{(-)} \\ \gamma_1 &= g_{(-)} f'_{(+)} f''_{(-)} + g_{(+)} f'_{(-)} g''_{(-)} \\ \delta_1 &= f_{(-)} f'_{(+)} f''_{(+)} + f_{(+)} f'_{(-)} g''_{(+)}, \end{aligned} \tag{15}$$

where we have used the *phase factors* $\Psi_{(\pm)}$ and $\Phi_{(\pm)}$ to define

$$\begin{aligned}
f_{(\pm)} &= \frac{e^{i\Psi_{(+)}(\theta)} \pm e^{i\Psi_{(-)}(\theta)}}{2} \\
f'_{(\pm)} &= \frac{e^{i\Psi_{(+)}(\theta')} \pm e^{i\Psi_{(-)}(\theta')}}{2} \\
f''_{(\pm)} &= \frac{e^{i\Psi_{(+)}(\theta+\theta')} \pm e^{i\Psi_{(-)}(\theta+\theta')}}{2} \\
g_{(\pm)} &= \frac{e^{i\Phi_{(+)}(\theta)} \pm e^{i\Phi_{(-)}(\theta)}}{2} \\
g'_{(\pm)} &= \frac{e^{i\Phi_{(+)}(\theta')} \pm e^{i\Phi_{(-)}(\theta')}}{2} \\
g''_{(\pm)} &= \frac{e^{i\Phi_{(+)}(\theta+\theta')} \pm e^{i\Phi_{(-)}(\theta+\theta')}}{2},
\end{aligned} \tag{16}$$

such that $(f, f', f'')_{(\pm)}$ correspond respectively to arguments $(\theta, \theta', (\theta + \theta'))$ with analogous notations for $(g, g', g'')_{(\pm)}$. Starting with (11) and tracing out the third index, one obtains the density matrix

$$\rho_{12} = \begin{vmatrix} \alpha_1\alpha_1^* & 0 & 0 & \alpha_1\delta_1^* \\ 0 & \beta_1\beta_1^* & \beta_1\gamma_1^* & 0 \\ 0 & \beta_1^*\gamma_1^* & \gamma_1\gamma_1^* & 0 \\ \alpha_1^*\delta_1 & 0 & 0 & \delta_1\delta_1^* \end{vmatrix}. \tag{17}$$

Defining

$$\tilde{\rho}_{12} = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \otimes \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \rho_{12}^* \otimes \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \otimes \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \tag{18}$$

one then obtains the matrix

$$\rho_{12}\tilde{\rho}_{12} = 2\begin{vmatrix} \alpha_1\alpha_1^*\delta_1\delta_1^* & 0 & 0 & \alpha_1^2\alpha_1^*\delta_1^* \\ 0 & \beta_1\beta_1^*\gamma_1\gamma_1^* & \beta_1^2\beta_1^*\gamma_1^* & 0 \\ 0 & \gamma_1^2\beta_1^*\gamma_1^* & \beta_1\beta_1^*\gamma_1\gamma_1^* & 0 \\ \delta_1^2\alpha_1^*\delta_1^* & 0 & 0 & \alpha_1\alpha_1^*\delta_1\delta_1^* \end{vmatrix}. \tag{19}$$

The matrix $(\rho_{12}\tilde{\rho}_{12})$ has the following eigenstates:

$$\begin{vmatrix} \frac{\alpha_1}{\delta_1} \\ 0 \\ 0 \\ 1 \end{matrix}\Bigg\rangle, \begin{vmatrix} \frac{\alpha_1}{\delta_1} \\ 0 \\ 0 \\ -1 \end{matrix}\Bigg\rangle, \begin{vmatrix} 0 \\ \frac{\beta_1}{\gamma_1} \\ 1 \\ 0 \end{matrix}\Bigg\rangle, \begin{vmatrix} 0 \\ \frac{\beta_1}{\gamma_1} \\ -1 \\ 0 \end{matrix}\Bigg\rangle \tag{20}$$

with the eigenvalues $4\alpha_1\alpha_1^*\delta_1\delta_1^*, 0, 4\beta_1\beta_1^*\gamma_1\gamma_1^*, 0$, respectively. Implementing the results of [8] (as in [5]), the 3-tangle is obtained as

$$\tau_{123} = 16(\alpha_1\alpha_1^*\beta_1\beta_1^*\gamma_1\gamma_1^*\delta_1\delta_1^*)^{\frac{1}{2}}, \tag{21}$$

noting that $\tau_{123} = 4(\text{product of non-zero roots})^{\frac{1}{2}}$, and using (20) to get the only two non-zero roots: $4\alpha_1\alpha_1^*\delta_1\delta_1^*$ and $4\beta_1\beta_1^*\gamma_1\gamma_1^*$. The 3-tangle is invariant under permutations of the subsystems (1, 2, 3).

Due to the unitarity of $\widehat{B}$ (after $\theta \to i\theta$) in (11) $\alpha_1\alpha_1^* + \beta_1\beta_1^* + \gamma_1\gamma_1^* + \delta_1\delta_1^* = 1$ and $0 \leqslant \tau_{123} \leqslant 1$. As the parameters (p, q, θ, θ') vary, the 3-tangle τ_{123} varies in the domain [0, 1]. The doubly periodic elliptic functions involved, expressed in terms of the

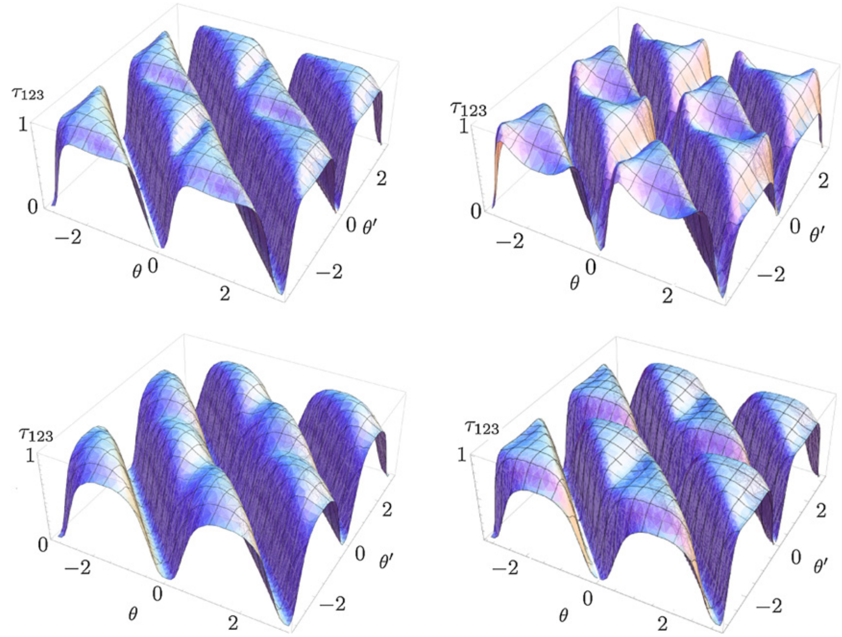

Figure 1. Variations of the 3-tangle τ_{123} as a function of (θ, θ'), by the action of $\widehat{B}$, on the product states $|111\rangle, |1\bar{1}\bar{1}\rangle, |\bar{1}1\bar{1}\rangle, |\bar{1}\bar{1}1\rangle$ in the subspace $V_{(e)}$ given by (11). The parameters $p = 0.1, q = 0.5$.

q-Pochhammer functions as in ratios (10), demand painstaking computations requiring rather involved algebra.

One can study entirely analogously $\widehat{B}(|1\bar{1}\bar{1}\rangle, |\bar{1}1\bar{1}\rangle, |\bar{1}\bar{1}1\rangle)$ in the subspace $V_{(e)}$ implementing respectively the sets of coefficients $(\alpha_i, \beta_i, \gamma_i, \delta_i)$, $i = 2, 3, 4$, as given by

$$\begin{aligned}
\alpha_2 &= f_{(+)}g'_{(+)}f''_{(-)} + f_{(-)}g'_{(-)}g''_{(-)},\\
\beta_2 &= g_{(+)}g'_{(+)}f''_{(+)} + g_{(-)}g'_{(-)}g''_{(+)},\\
\gamma_2 &= g_{(-)}g'_{(+)}f''_{(+)} + g_{(+)}g'_{(-)}g''_{(+)},\\
\delta_2 &= f_{(-)}g'_{(+)}f''_{(-)} + f_{(+)}g'_{(-)}g''_{(-)},\\
\alpha_3 &= f_{(-)}g'_{(+)}g''_{(-)} + f_{(+)}g'_{(-)}f''_{(-)},\\
\beta_3 &= g_{(-)}g'_{(+)}g''_{(+)} + g_{(+)}g'_{(-)}f''_{(+)},\\
\gamma_3 &= g_{(+)}g'_{(+)}g''_{(+)} + g_{(-)}g'_{(-)}f''_{(+)},\\
\delta_3 &= f_{(+)}g'_{(+)}g''_{(-)} + f_{(-)}g'_{(-)}f''_{(-)},\\
\alpha_4 &= f_{(-)}f'_{(+)}g''_{(+)} + f_{(+)}f'_{(-)}f''_{(+)},\\
\beta_4 &= g_{(-)}f'_{(+)}g''_{(-)} + g_{(+)}f'_{(-)}f''_{(-)},\\
\gamma_4 &= g_{(+)}f'_{(+)}g''_{(-)} + g_{(-)}f'_{(-)}f''_{(-)},\\
\delta_4 &= f_{(+)}f'_{(+)}g''_{(+)} + f_{(-)}f'_{(-)}f''_{(+)}.
\end{aligned} \tag{22}$$

Figure 1 shows the rich structure with subtle variations for τ_{123} by the action of $\widehat{B}$ on the product states $|111\rangle, |1\bar{1}\bar{1}\rangle, |\bar{1}1\bar{1}\rangle, |\bar{1}\bar{1}1\rangle$ in the subspace $V_{(e)}$. This rich structure is indeed the

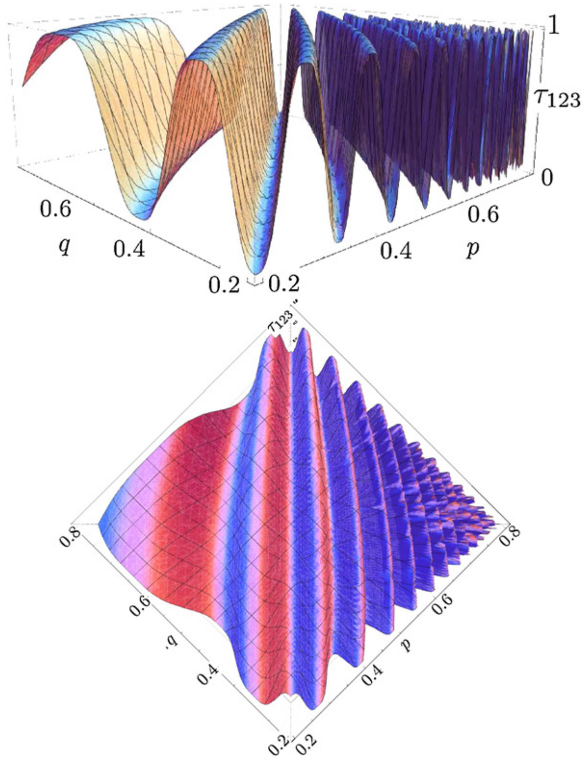

Figure 2. Variations (cross-sectional and top views) of the 3-tangle τ_{123} as a function of (p, q) for $(\theta = \pi/3, \theta' = \pi/6)$, by the action of $\widehat{B}$ on the product state $|111\rangle$.

real attraction of the unitarized 8-vertex case. We note that for $(\theta + \theta') = 0$, we have $\tau_{123} = 0$, so that in the domain $(-\pi, \pi)$ for both (θ, θ'), there are diagonal lines of symmetry, with a line of zero value passing through the origin. One further notes that $(\Psi, \Phi)_\pm \to (\Psi, \Phi)_\pm$ and hence $(\tau_{123} \to \tau_{123})$ for $p \to 1/p$ (and also with $q \to 1/q, \theta \to -\theta$). There are more intricate and subtle lines of symmetry as evident in figure 2, where we show the oscillations of τ_{123} between zero and unity, as a function of (p, q) for $\widehat{B}|111\rangle$. The results for the other subspace $V_{(o)}$, namely $\widehat{B}(|\bar{1}\bar{1}\bar{1}\rangle, |\bar{1}11\rangle|1\bar{1}1\rangle, |11\bar{1}\rangle)$, follows from the symmetry of $V_{(e)}$ and $V_{(o)}$ under the action of $\widehat{B}$ as stated in (13) and (14). Combining these results one can then study the action of $\widehat{B}$ on the general product state, namely

$$\widehat{B}\{(x_1|1\rangle + x_{\bar{1}}|\bar{1}\rangle) \otimes (y_1|1\rangle + y_{\bar{1}}|\bar{1}\rangle) \otimes (z_1|1\rangle + z_{\bar{1}}|\bar{1}\rangle)\}, \tag{23}$$

with some more straightforward algebra.

If we write

$$\begin{aligned}\widehat{B}\{(x_1|1\rangle + x_{\bar{1}}|\bar{1}\rangle) \otimes (y_1|1\rangle + y_{\bar{1}}|\bar{1}\rangle) \otimes (z_1|1\rangle + z_{\bar{1}}|\bar{1}\rangle)\}\\ = c_{111}|111\rangle + c_{1\bar{1}\bar{1}}|1\bar{1}\bar{1}\rangle + c_{\bar{1}1\bar{1}}|\bar{1}1\bar{1}\rangle + c_{\bar{1}\bar{1}1}|\bar{1}\bar{1}1\rangle\\ + c_{\bar{1}\bar{1}\bar{1}}|\bar{1}\bar{1}\bar{1}\rangle + c_{\bar{1}11}|\bar{1}11\rangle + c_{1\bar{1}1}|1\bar{1}1\rangle + c_{11\bar{1}}|11\bar{1}\rangle\end{aligned} \tag{24}$$

and use the definitions from equation (15), equation (16) and equation (22), then we have for example

$$c_{111} = (x_1 y_1 z_1)\alpha_1 + (x_1 y_{\bar{1}} z_{\bar{1}})\alpha_2 + (x_{\bar{1}} y_1 z_{\bar{1}})\alpha_3 + (x_{\bar{1}} y_{\bar{1}} z_1)\alpha_4. \tag{25}$$

Similarly, $(c_{1\bar{1}\bar{1}}, c_{\bar{1}1\bar{1}}, c_{\bar{1}\bar{1}1})$ are obtained respectively by substituting $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in c_{111}, the coefficients $(\beta_1, \beta_2, \beta_3, \beta_4)$, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ and $(\delta_1, \delta_2, \delta_3, \delta_4)$. Next

$$(c_{111}, c_{1\bar{1}\bar{1}}, c_{\bar{1}1\bar{1}}, c_{\bar{1}\bar{1}1}) \leftrightharpoons (c_{\bar{1}\bar{1}\bar{1}}, c_{\bar{1}11}, c_{1\bar{1}1}, c_{11\bar{1}})$$

can be shown to correspond to

$$(x_1 y_1 z_1) \leftrightharpoons (x_{\bar{1}} y_{\bar{1}} z_{\bar{1}}).$$

Thus, one has eight coefficients (c_{ijk}) involving

(i) six *complex* parameters (with one constraint for the overall normalization), and
(ii) four *real* parameters (p, q, θ, θ'), as indicated in equation (10) and equation (16).

Thus, there are (2.6−1+4)=15 parameters. Varying all these parameters suitably, one can obtain amongst others, some special (well-known) states. Thus, for

$$c_{111} = \pm c_{\bar{1}\bar{1}\bar{1}} = \frac{1}{\sqrt{2}} \tag{26}$$

and the other six c's vanishing, one obtains the $|\mathrm{GHZ}\rangle$ states.

Again, for example, for

$$c_{1\bar{1}\bar{1}} = c_{\bar{1}1\bar{1}} = c_{\bar{1}\bar{1}1} = \frac{1}{\sqrt{3}} \tag{27}$$

and the other c's vanishing, or for

$$c_{\bar{1}11} = c_{1\bar{1}1} = c_{11\bar{1}} = \frac{1}{\sqrt{3}} \tag{28}$$

and the other c's vanishing, one obtains the $|\mathrm{W}\rangle$ states.

Obtaining explicitly the points in the multi-dimensional parameter space corresponding to specific states is beyond the scope of this communication.

3. Conclusions and outlook

In summary, we have studied quantum entanglements induced on product states by the action of 8-vertex braid matrices, rendered unitary with purely imaginary spectral parameters (rapidity). First, the unitarity was displayed via the 'canonical factorization' of the coefficients of the projectors spanning the basis for the simpler 6-vertex model, as well as the more complex 8-vertex model. For the 8-vertex model, we have further analyzed and computed the action of the braid matrix $\widehat{B}$, on basic pure product states to see whether there is indeed quantum entanglement after unitarization. We found a conclusive answer: quantum entanglement was indeed generated. This adds one more *new* facet to the famous and fascinating features of the 8-vertex model. Using the density matrix, we then extracted the 3-tangle explicitly for this complex situation and obtained a measure of the 3-particle correlation. The double periodicity and analytic properties of the elliptic functions involved led to a rich structure of the 3-tangle, quantifying the entanglement. We thus explored the complex relationship between topological and quantum entanglement.

We list the essential features of our study, along with an outlook as follows.

(i) Some authors [11] had studied how *local* unitary transformations, conserving the entanglement measures (2-tangles and 3-tangles), can be implemented to classify entangled states as unitary transforms of standard ones, such as $|\mathrm{GHZ}\rangle$ and $|\mathrm{W}\rangle$. Here we showed that *non-local* unitary action of unitary braid operators can lead to unified views of *all* 3-particle entanglements situating them in the domains of the variable parameters corresponding to the class of the braid operator implemented. Thus, the study here provides a broader and deeper understanding of the entanglement landscape. We intend to study further such aspects thoroughly elsewhere.

(ii) In [5], two classes of unitary braid operators were implemented to generate parametrized 3-tangles. As emphasized in the title, our goal was to bridge two fascinating domains of profound significance—namely topological and quantum entanglements. In this communication, we turned to the celebrated 8-vertex model. The fascinating aspects of the statistical models with phase transitions (for the real braid matrix) are well known. Here, we unitarized it via the simple passage to imaginary rapidity ($\theta \to \mathrm{i}\theta$). This displayed the novel possibility of the complex 8-vertex braid operator. It led to the study of the 3-tangles [8] parametrized by sums of products of ratios of the q-Pochhammer functions. The figures provided some snapshots of the rich landscape. This certainly needs further detailed explorations.

(iii) The role of our 'canonical factorization' [7] in displaying passage to unitarity under $\theta \to \mathrm{i}\theta$ has already been stressed and illustrated. Such unitarization has broader prospects. Different classes (not only, say, 6-vertex or 8-vertex) of braid operators can provide the starting points. We intend to explore, from this point of view, a new class of braid operators in a future article. Remarkable and surprising properties of the new class of braid matrices ($S\hat{O}(N)$ and $S\hat{p}(N)$) have been pointed out [12]. We will show elsewhere how they lead to Temperley–Lieb algebra, spin-chains and quantum entanglements. The techniques introduced here will lead to broader possibilities.

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