Threshold-induced phase transition in kinetic exchange models

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We study an ideal-gas-like model where the particles exchange energy stochastically, through energy-conserving scattering processes, which take place *if and only if* at least one of the two particles has energy below a certain energy threshold (interactions are initiated by such low-energy particles). This model has an intriguing phase transition in the sense that there is a critical value of the energy threshold below which the number of particles in the steady state goes to zero, and above which the average number of particles in the steady state is nonzero. This phase transition is associated with standard features like "critical slowing down" and nontrivial values of some critical exponents characterizing the variation of thermodynamic quantities near the threshold energy. The features are exhibited not only in the mean-field version but also in the lattice versions.

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I. INTRODUCTION

The kinetic theory of gases played a pivotal role in the development of statistical mechanics, which is more than a century old. This theory describes a gas as a collection of a large number of particles (atoms or molecules) which are constantly in random motion, and these rapidly moving particles constantly collide with each other and exchange energy. In the ideal gas, this energy is only kinetic. Recently physicists have been studying two-body kinetic exchange models in socioeconomic contexts in the rapidly growing interdisciplinary fields of sociophysics [1] and econophysics [2]. The two-body exchange dynamics has been developed in the context of modeling income, money, or wealth distributions in a society [3–10] and modeling opinion formation in the society [11–14], analogous to the kinetic theory model of ideal gases. These studies have given deeper insights into and different perspectives on the simple physics of two-body kinetic exchange dynamics. In this context of wealth exchange processes, Iglesias and co-workers [15,16] considered a model for the economy where the poorest in the society (atom with least energy in the gas) at any stage takes the initiative to go for a trade (random wealth or energy exchange) with anyone else. Interestingly, in the steady state, one obtained a self-organized poverty line, below which no one could be found and above which a standard exponential decay of the distribution (Gibbs) was obtained.

Here, we study a model where N particles, interact among themselves through two-body energy- (x-) conserving stochastic scatterings with at least one of the particles having energy below a threshold θ (poverty line in the equivalent economic model). The states of particles are characterized by the energy $\{x_i\}$, $i = 1,2,\ldots,N$, such that $x_i > 0$, $\forall\, i$ and the total energy $E = \sum_i x_i$ is conserved ($= N$ here, such that the average energy of the system $\bar{E} = E/N = 1$). The evolution of the system is carried out according to the following dynamics:

$$
\begin{aligned}
x_i^{<\prime} &= \epsilon(x_i^< + x_j),\\
x_j' &= (1-\epsilon)(x_i^< + x_j),
\end{aligned}
\tag{1}
$$

where $x_i^< < \theta$ (threshold energy or " poverty line") and ϵ $(0 \leqslant \epsilon \leqslant 1)$ is a stochastic variable, changing with time (scattering). It can be seen that the quantity x is conserved during each collision: $x_i^{<\prime} + x_j' = x_i^< + x_j$. The question of interest is "What is the steady state distribution $p(x)$ of energy x in such systems?"

In the standard case, when the threshold energy goes to infinity ($\theta \to \infty$), we know that the steady state energy distribution will be the exponential Gibbs distribution [$p(x) \sim \exp(-x)$] [3]. However, when a finite threshold energy is introduced ($\theta > 0$), several new and intriguing features appear. These features are exhibited not only in the mean-field version (with infinite-range interaction, pairs of particles randomly chosen from N particles) but also in the lattice versions (with nearest neighbor interactions, i.e., exchanges between the nearest neighbors on lattice sites).

II. MODEL SIMULATIONS AND RESULTS

A. The model

We simulate a system of N particles (agents). At any time t, we select randomly a particle i. If the energy of the particle is below a prescribed threshold energy θ, then

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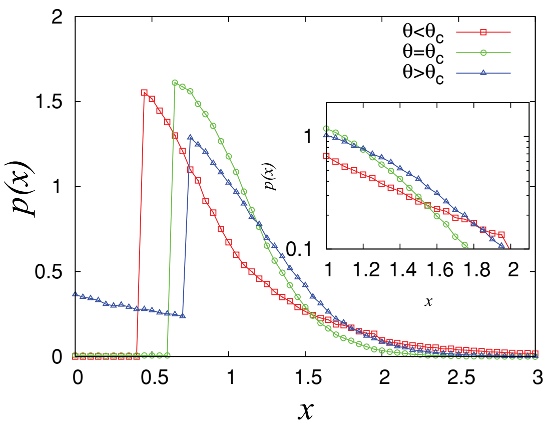

FIG. 1. (Color online) Energy distribution $p(x)$ in the steady state ($t > \tau$), for different θ values. The inset shows a semi-log plot of the energy distribution. The tail of the distribution is Gibbs-like ($N = 10^5$; mean-field model with average taken over many independent initial conditions).

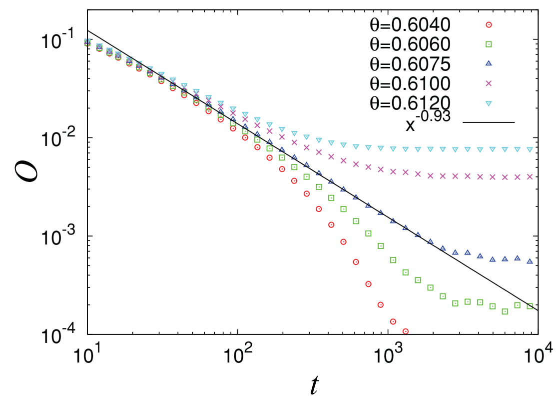

FIG. 3. (Color online) Variations of O versus time t, shown for different θ values. At the critical value θ_c, the order parameter follows a power law decay with exponent $\delta \simeq 0.93$ (shown by the solid line). ($N = 10^7$; mean-field model with average taken over many independent initial conditions.)

it collides with any other random particle j (in the mean-field model) which can have any energy whatsoever, and the two particles will exchange energy according to the Gibbs-Boltzmann dynamics of Eq. (1). After each such successful collision, the time is incremented by unity. The dynamics will continue for an indefinite period, unless there is no particle left below the threshold energy, in which case the dynamics will freeze. If the dynamics gets frozen (when $x_i > \theta$ for all i), we employ a "mild" perturbation such that a randomly chosen particle will be dropped to the lower level ($< \theta$) by giving up its energy to anyone else (to ensure total energy conservation). It can be shown that the addition of this perturbation does not alter the relevant quantities for a thermodynamically large system and simply ensures ergodicity in the system. After sufficiently large time $t > \tau$, a steady state is reached when the energy distribution $p(x)$ (and also other average quantities) do not change with time. We start with different initial random configurations, where the states of particles are characterized by the energies $\{x_i\}$, $i = 1,2,\ldots,N$, which are drawn randomly from a uniform distribution such that $x_i > 0$, $\forall$ i and the average energy $\bar{E} = \sum_i x_i/N$ is set to unity. We find the system to be *ergodic* [the steady state distribution $p(x)$ is *independent* of the initial conditions $\{x_i\}$], and we take steady state averages over all such independent initial conditions to evaluate the quantities of interest.

We study mainly three cases: (a) the mean-field (MF) (or infinite-range) case where i and j in Eq. (1) can represent any two particles or agents in the system; (b) the one-dimensional case where $j = i \pm 1$ along a chain; and (c) the two-dimensional (2D) case, where $j = i \pm \delta$ with δ representing neighbors of i. In our studies we consider a 2D square lattice.

We observe that for finite values of the energy threshold θ, the steady state energy distribution is no longer the simple Gibbs-Boltzmann distribution. We also find that O [$\equiv \int_0^\theta p(x)dx$], the average number of particles below the threshold energy in the steady state, is zero for θ values below or at a critical threshold energy θ_c, and for $\theta > \theta_c$, O is nonzero. The steady state value of O, the average number of particles below the threshold energy θ, is seen to act like an "order parameter" of the system. We study the relaxation

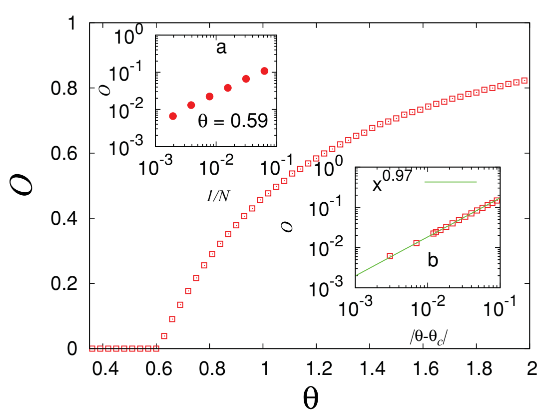

FIG. 2. (Color online) Simulation results for the variation of O, the average number of particles below the threshold energy θ in the steady state ($t > \tau$), against threshold energy θ. Inset: (a) Result $O \to 0$ for $\theta = 0.59$ ($< \theta_c$) as $N \to \infty$; (b) scaling fit $(\theta - \theta_c)^\beta$ with $\beta \simeq 0.97$. ($N = 10^5$; mean-field model with average taken over many independent initial conditions.)

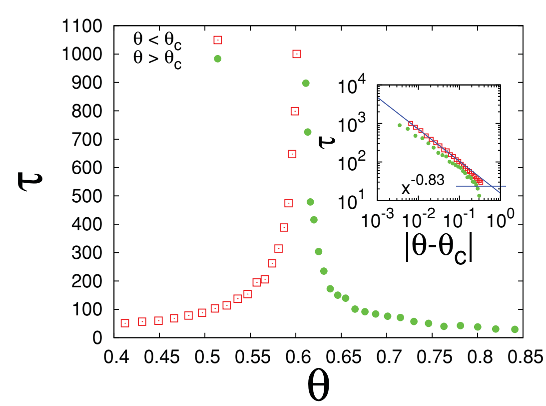

FIG. 4. (Color online) Variation of τ versus θ. Inset: Scaling fit $\tau \sim |\theta - \theta_c|^{-z}$, with exponent $z \simeq 0.83$ (mean-field case; $N = 10^5$).

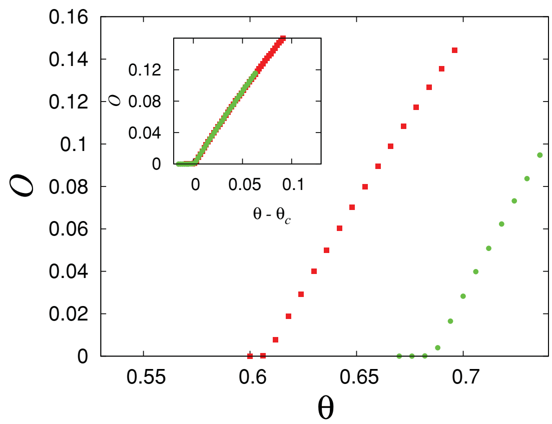

FIG. 5. (Color online) Variation of steady state order parameter $O(\theta)$ against θ for dynamics following Eq. (1) (denoted by red squares) and Eq. (2) (denoted by green circles). Inset: O vs $(\theta - \theta_c)$ for both cases. ($N = 10^5$; mean-field case.)

dynamics in the system: the relaxation of $O(t)$ to the steady state value of O [$= O(\theta)$ for $t > \tau(\theta)$, the "relaxation time"]. We find that $\tau(\theta)$ grows as θ approaches θ_c, and eventually diverges at θ_c. The details of the results are given below.

B. Results: Mean-field model

In the mean-field (long-range) model, we first look for any particle (i) with energy $x_i < \theta$ and then this particle is allowed to interact with any other particle (j), following Eq. (1). This continues until either the steady state or a frozen state with $x_i > \theta$ for all i is reached. In the case of a frozen state, as mentioned earlier, any one particle is picked up randomly and it loses its energy to any other (randomly chosen) particle, and goes below the threshold θ. This induces further dynamics. Eventually a steady state is reached. We study this steady state energy distribution $p(x)$ (see Fig. 1), and the order parameter $O \equiv \int_0^\theta p(x)dx$ (see Fig. 2), showing a "phase transition" at $\theta_c \simeq 0.607 \pm 0.001$. A power law fit $O \sim (\theta - \theta_c)^\beta$ gives $\beta \simeq 0.97 \pm 0.01$.

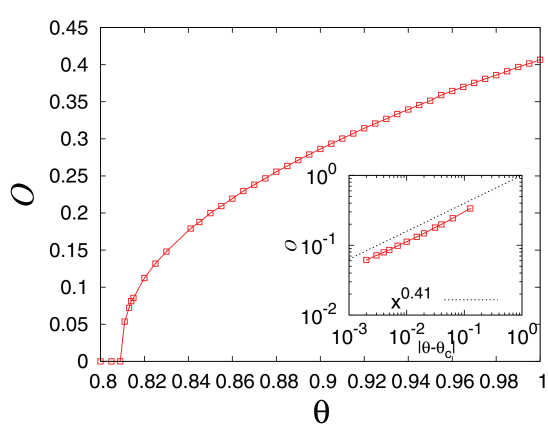

FIG. 6. (Color online) Variation of O, the average number of particles below the threshold energy θ in the steady state ($t > \tau$), against threshold energy θ, following dynamics of Eq. (1) for 1D ($N = 10^4$). Inset: Scaling fit $(\theta - \theta_c)^\beta$ with $\beta \simeq 0.41$.

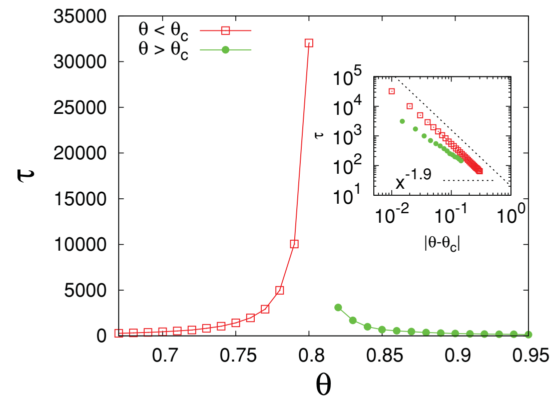

FIG. 7. (Color online) Relaxation time τ as a funtion of θ. Clearly τ diverges as $\theta \to \theta_c$. Inset: Numerical fit to $\tau \sim |\theta - \theta_c|^{-z}$, with $z \simeq 1.9$. ($N = 10^4$, 1D case.)

We also studied the relaxation behavior of O. At $\theta = \theta_c$, the $O(t)$ variation fits well with $t^{-\delta}$; $\delta \simeq 0.93 \pm 0.01$ (see Fig. 3). The relaxation time τ is estimated numerically from the time value at which O first touches the steady state value $O(\theta)$ within a preassigned error limit. We find diverging growth of the relaxation time τ near $\theta = \theta_c$ (see Fig. 4), showing "critical slowing down" at the critical value θ_c. The values of the exponent z for the divergence in $\tau \sim |\theta - \theta_c|^{-z}$ have been estimated (for both $\theta > \theta_c$ and $\theta < \theta_c$). For the mean-field model, the fitting value for the exponent $z \simeq 0.83 \pm 0.01$.

We have also studied the universality of this behavior by generalizing the dynamics in Eq. (1) to

$$\begin{aligned} x_i^{<\prime} &= \epsilon_1 x_i^< + \epsilon_2 x_j, \\ x_j' &= (1 - \epsilon_1) x_i^< + (1 - \epsilon_2) x_j, \end{aligned} \tag{2}$$

where ϵ_1 and ϵ_2 are random stochastic variables within the range [0,1]. The critical point θ_c shifts to $\theta_c \simeq 0.69$ ($\theta_c \simeq 0.61$ for $\epsilon_1 = \epsilon_2 = \epsilon$). The transition behavior is seen to be universal near the critical point θ_c, but the critical point depends specifically on the model (see Fig. 5).

C. Results: One-dimensional model

In the one-dimensional lattice version, the particles are arranged on a periodic chain. At any time t, we randomly select a lattice site i. If the energy of the corresponding particle is below a prescribed threshold energy θ, then it collides with any

TABLE I. Comparison of critical exponents of this model with those of the Manna model [18].

		This model	Manna model
β	1D	0.41 ± 0.02	0.382 ± 0.019
	2D	0.67 ± 0.01	0.639 ± 0.009
	MF	0.97 ± 0.01	1
z	1D	1.9 ± 0.05	1.876 ± 0.135
	2D	1.2 ± 0.01	1.22 ± 0.029
	MF	0.83 ± 0.01	1
δ	1D	0.19 ± 0.01	0.141 ± 0.024
	2D	0.43 ± 0.02	0.419 ± 0.015
	MF	0.93 ± 0.01	1

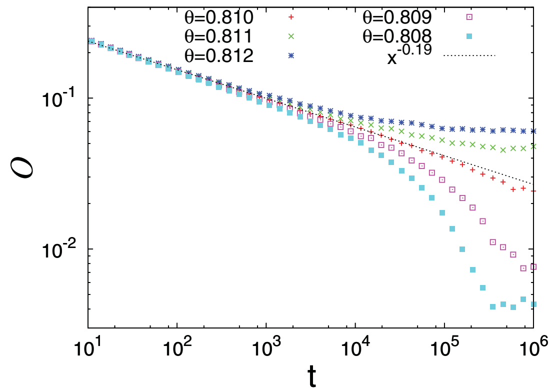

FIG. 8. (Color online) Variation of $O(t)$ versus t for different θ values for 1D case ($N = 10^4$).

one randomly chosen nearest neighbor j ($= i \pm 1$) which can have any energy whatsoever, and the two particles exchange energy according to Eq. (1). After each such successful collision, the time is incremented by unity. This process is continued until steady state is reached. The steady state order parameter O variations against theshold θ is shown in Fig. 6, with exponent $\beta \simeq 0.41 \pm 0.02$ and $\theta_c \simeq 0.810 \pm 0.001$. The fitting value for exponent z turn out to be around 1.9 ± 0.05 (see Fig. 7). Also we find $\delta \simeq 0.19 \pm 0.01$ (see Fig. 8). It may be noted that a related chain model with such an energy cutoff for kinetics, where the effective temperature is varied, has been studied [17]. Though the behavior is similar, the effective critical behavior (exponent values) seems to be quite different.

D. Results: Two-dimensional model

For the 2D lattice version, the particles are arranged on a square lattice, and this time one of the four nearest neighbors of i is chosen randomly as particle j. If the energy of the particle is below a prescribed threshold energy θ, then it collides with any one randomly chosen nearest neighbor j which can have any energy whatsoever, and the two particles exchange energy according to Eq. (1). After each such successful collision,

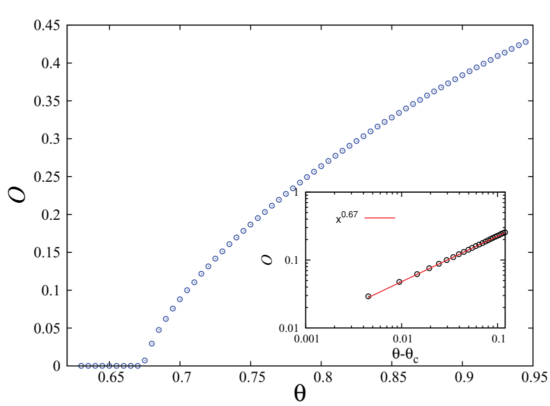

FIG. 9. (Color online) Variation of O, the average number of particles below the threshold energy θ in the steady state ($t > \tau$), against threshold energy θ, following dynamics of Eq. (1) for $2D$ case. The simulation is done for lattice size 1000×1000. Inset: Scaling fit $(\theta - \theta_c)^\beta$ with $\beta \simeq 0.67$.

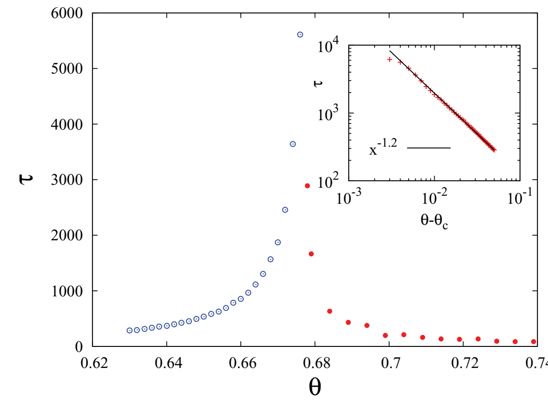

FIG. 10. (Color online) Relaxation time (τ), showing divergence as θ approches θ_c from both sides. Inset: Numerical fit to $\tau \sim |\theta - \theta_c|^{-z}$ for $\theta < \theta_c$ ($z \simeq 1.2 \pm 0.01$). (Simulations for 100×100 system, 2D case.)

the time is incremented by unity. This process is continued until steady state is reached. Variation of the steady state order parameter O against theshold θ is shown in Fig. 9, with exponent $\beta \simeq 0.67 \pm 0.01$ and $\theta_c \simeq 0.675 \pm 0.005$. Also, we find $z \simeq 1.2 \pm 0.01$ (see Fig. 10) and $\delta \simeq 0.43 \pm 0.02$ (see Fig. 11).

All these estimated values of the critical exponents β, z, and δ are summarized in Table I.

III. FINITE SIZE EFFECT

We have also studied the time variation of O at different sizes (N) at θ_c. Plots of $O(t)t^{\delta}$ as a function of t/N^{σ} for different values of the system size N (at the critical point) are expected to collapse on a single curve. However, as we have used a special dynamics which never allows the system to fall into the absorbing state, in this case the activity saturates at a small steady value (see Fig. 12) instead of showing the finite size cutoff.

To study finite size effects in the decay of activity at the critical point, one has to remove the perturbation and allow the system to be trapped in the absorbing states. Using this

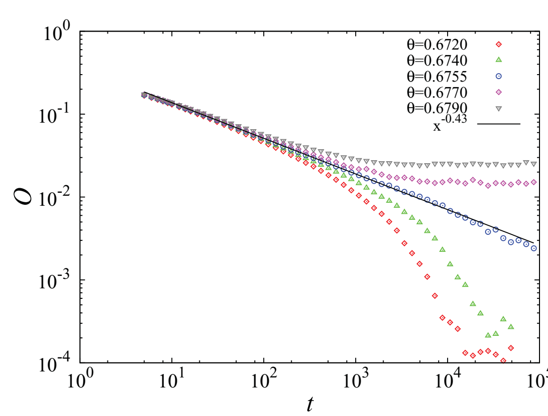

FIG. 11. (Color online) Variation of $O(t)$ versus time t, shown for different θ values for the 2D case. At the critical value θ_c, the order parameter follows a power law decay with exponent $\delta \simeq 0.43$. The simulation is done for lattice size 500×500.

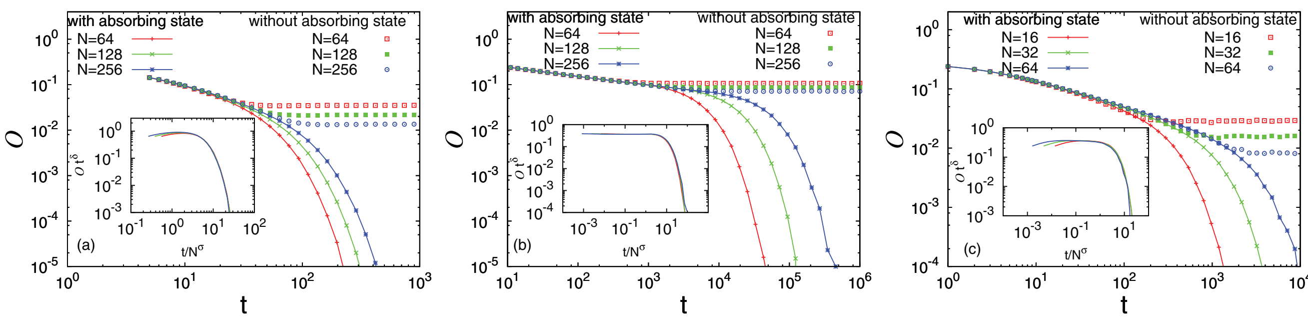

FIG. 12. (Color online) Study of finite size effect: The variation of $O(t)$, versus t at $\theta = \theta_c$ for systems of different sizes N. The plots with bare symbols correspond to the case when there is no absorbing state and those with symbols connected by solid lines correspond to the presence of absorbing state. (a), (b), and (c) correspond to MF, 1D, and 2D, respectively. Inset: Ot^δ vs t/N^σ in the presence of an absorbing state, for different system sizes N, showing collapse onto a single curve for $\delta = 0.93$, 0.19, and 0.43 and $\sigma = 0.53$, 1.53, and 1.55 for MF, 1D, and 2D, respectively.

dynamics, we have studied the effects of finite system size. Figure 12 shows the decay of $O(t)$ with t at the critical point for MF, 1D, and 2D systems, respectively. The insets in the corresponding figures show the data collapse. The fitting values of the exponent σ are $\sigma = 0.53 \pm 0.02, 1.53 \pm 0.05$, and 1.55 ± 0.05 for MF, 1D, and 2D, respectively, with δ values given in Table I.

IV. SUMMARY AND DISCUSSION

Inspired by the success of the kinetic exchange models of market dynamics (see, e.g., [2–4]) and the observation that the poor or economically backward in the society take the major initiative in the market dynamics (see, e.g., [15,16] and also [19,20]), we consider an ideal-gas-like model of a gas (or market) where at leat one of the particles (or agents) has energy (money) x less than a threshold (poverty line) value θ take the initiative to scatter (trade) with any other particle (agent) in the system, following energy- (money-) conserving random processes [following Eq. (1)]. For $\theta \to \infty$, the model reduces to the kinetic model of an ideal gas with Gibbs distribution. The steady state is found to be ergodic (steady state results are independent of initial conditions). The perturbation employed in the frozen cases ($x_i > \theta$ for all i) also does not affect significantly the thermodynamic quantities [e.g., the steady state value of O for $\theta < \theta_c$ goes to 0 with $1/N$, as can be seen from inset (a) of Fig. 2].

In general, we find that the steady state distribution $p(x)$ (see Fig. 1 for the mean field, where each particle can interact irrespective of its distance from the active particle or agent having energy or money less than the threshold or poverty line θ) differs in form significantly from the Gibbs distribution, for finite values of θ. The order parameter O, giving the average fraction of particles (or agents) having energy (or money) below θ, shows a phase transition behavior: $O = 0$ for $\theta < \theta_c$ and $O \neq 0$ for $\theta > \theta_c$ (see Fig. 2 for the mean field, Fig. 6 for 1D, and Fig. 9 for 2D, respectively). The critical values are given by $\theta_c \simeq 0.61$, 0.81, and 0.68 for mean-field, 1D, and 2D cases, respectively. The variation of O near θ_c is quite universal (see Fig. 5). We find $O \sim (\theta - \theta_c)^\beta$ with $\beta \simeq 0.97$, 0.41, and 0.67 for mean-field, 1D, and 2D cases, respectively. We also find that the relaxation time τ diverges strongly near θ_c as $\tau \sim (\theta - \theta_c)^{-z}$ with $z \simeq 0.83$, 1.9, and 1.2 for mean-field, 1D, and 2D cases, respectively. Finally, at $\theta = \theta_c$, $O(t) \sim t^{-\delta}$ where $\delta \simeq 0.93$, 0.19, and 0.43 for mean field, 1D, and 2D cases, respectively (see Table I).

It might be mentioned here that the above exponent values are indeed very close to those of the Manna universality (MU) class ([18]; see also [21,22]) in the mean-field, 1D, 2D cases: Our estimates of $\beta \simeq 0.97, 0.41$, and 0.67 for mean-field, 1D, and 2D cases, respectively, are quite close to $\beta \simeq 1, 0.38$, and 0.64 for the corresponding MU cases; $\delta \simeq 0.93, 0.19$, and 0.43 for mean-field, 1D, and 2D cases are also close to $\delta \simeq 1, 0.14$, and 0.42 in the corresponding MU cases. However, it may be noted that significant differences in the above estimates do exist. Also, $z \simeq 0.83$, 1.9, and 1.2 for mean-field, 1D, and 2D cases might be compared with $z \simeq 1$, 1.87, and 1.22 in the corresponding MU cases. These discrepancies could be due to the finite size effect, and in that case the critical behavior of our model would belong to the MU class. As one can see, the estimated values of the exponents β, δ, and z fit reasonably with the scaling relation $\delta = \beta/z$ within our limits of accuracy. In this connection, it is worth mentioning that the violation of the above scaling relation has also been observed [23], although such discrepancies seem to get removed if one uses all-sample averages instead of averages over surviving samples [24]. In our case, however, this scaling relation seems to hold, as our simulation results correspond to all-sample averages.

In summary, when the energy threshold θ is introduced in the kinetic theory of the ideal gas such that stochastic energy-conserving scatterings between any two particles can take place only when at least one has energy less than θ, the gas system shows an intriguing dynamic phase transition at $\theta = \theta_c$, having the exponent values in the mean field (long-range scattering exchange), one dimension, and two dimensions, as estimated here using Monte Carlo simulation, as given in Table I.

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