JOURNAL OF MATHEMATICAL PHYSICS **54**, 013517 (2013)

New classes of spin chains from $(S\widehat{O}_{(q)}(N), S\widehat{p}_{(q)}(N))$ Temperley-Lieb algebras: Data transmission and (q, N) parametrized entanglement entropies

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(Received 14 August 2012; accepted 17 December 2012; published online 29 January 2013)

A Temperley-Lieb algebra is extracted from the operator structure of a new class of $N^2 \times N^2$ braid matrices presented and studied in previous papers and designated as $S\widehat{O}_{(q)}(N)$, $S\widehat{p}_{(q)}(N)$ for the q-deformed orthogonal and symplectic cases, respectively. Spin chain Hamiltonians are derived from such braid matrices and the corresponding chains are studied. Time evolutions of the chains and the possibility of transition of data encoded in the parameters of mixed states from one end to the other are analyzed. The entanglement entropies $S(q, N)$ of eigenstates of the crucial operator, namely, the q-dependent $N^2 \times N^2$ projector P_0 appearing in the corresponding Hamiltonian are obtained. Study of entanglements generated under the actions of $S\widehat{O}(N)$, $S\widehat{p}(N)$ braid operators, unitarized with imaginary rapidities (spectral parameters) is presented as a perspective. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4774211]

I. INTRODUCTION

Interesting developments in fault-tolerant quantum computation using the braiding of anyons have brought the theory of braid groups at the very core of topological quantum computing.[1] Moreover, the recent study by Kauffman and Lomonaco that the "Bell matrix," a specific braiding operator from the solution of the Yang-Baxter equation, is *universal* implies that in principle all quantum gates can be constructed from braiding operators together with single qubit gates.[2] In another very recent paper, the authors presented a new class of braiding operators from the Temperley-Lieb algebra[3] that generalized the Bell matrix to multi-qubit systems, thus unifying the Hadamard and Bell matrices within the same framework.[4]

Here, we extract the Temperley-Lieb algebra from the operator structure of a new class of $N^2 \times N^2$ braid matrices presented and studied in previous papers and designated as $S\widehat{O}_{(q)}(N)$, $S\widehat{p}_{(q)}(N)$ for the q-deformed orthogonal and symplectic cases, respectively.[5] The connection between spin chains and Temperley-Lieb algebras is well-established.[6] We derive the spin chain Hamiltonians from such braid matrices and study the corresponding chains. We then analyze the time evolutions of the chains and the possibility of transition of data encoded in the parameters of mixed states from one end to the other, following studies such as Ref. 7. We further obtain the entanglement entropies $S(q, N)$ of eigenstates of the crucial operator, namely, the q-dependent $N^2 \times N^2$ projector P_0 appearing in the corresponding Hamiltonian.

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In a previous paper,[5] the central result was the "canonical factorization" of the coefficient of each projector (see Eq. (1) below). The spectral parameter θ in the coefficients is called "rapidity."[8,9] The "rapidity" parameter Yang-Baxterizes the braid matrix.[10–12] Usually it is real, as are the elements of each projector. But under the transition $\theta \to (i\theta)$, each coefficient due to canonical factorization becomes a "phase." Under complex conjugation $(i\theta \to -i\theta)$, each coefficient is inverted. Hence, if the projector(s) involved are real and also symmetric (as P_0 in this paper, or the projectors of the 8-vertex braid matrix case studied in Ref. 13), quite evidently $\widehat{R}(\theta)$ becomes *unitary*. This is what we called "unitarization with imaginary rapidity," and we presented a general formulation of this approach in Ref. 13. Such a braid matrix, acting on the direct product of any three states in the base space induces a unitary transformation preserving amplitudes. If such actions further generate states exhibiting "entanglement" (see explicit treatments of Ref. 13), we call them "entanglement generated by braid matrices unitarized with imaginary rapidities." In the 8-vertex braid matrix,[13] we displayed the remarkable and attractive properties of entanglement thus induced. In this paper, we present as a perspective that we can unitarize with imaginary rapidities as in Ref. 13, the braid operators $S\widehat{O}(N)$, $S\widehat{p}(N)$, and then study the entanglements generated under their actions.

II. BRAID MATRIX FORMALISM

In the context of exhaustive construction of "canonically factorized" forms [5] of braid matrices, new classes of solutions with remarkable and intriguing properties were obtained for $SO_q(N)$ and $Sp_q(N)$. To distinguish these special cases, they were, respectively, denoted by $S\widehat{O}_q(N)$ and $S\widehat{p}_q(N)$. The aspects of that formalism[5] necessary for our present study are summarized below.

For $SO_q(N)$ and $Sp_q(N)$, the standard cases[14] were first expressed in the form:

$$\widehat{R}(\theta) = \frac{f_+(-\theta)}{f_+(\theta)} P_+ + \frac{f_-(-\theta)}{f_-(\theta)} P_- + \frac{f_0(-\theta)}{f_0(\theta)} P_0, \tag{1}$$

where (P_+, P_-, P_0) form a *complete basis* of projectors, satisfying

$$P_i P_j = \delta_{ij} P_i, \quad P_+ + P_- + P_0 = I. \tag{2}$$

Since these are $N^2 \times N^2$ q-dependent matrices, so is $\widehat{R}(\theta)$. All θ-dependence of $\widehat{R}(\theta)$ is in the coefficients $(f_+(\theta), f_-(\theta), f_0(\theta))$. As consequence of Eq. (2)

$$\widehat{R}(\theta)\,\widehat{R}(-\theta) = I_{(N^2\times N^2)} = I_N \otimes I_N \equiv I \otimes I, \tag{3}$$

where I is the $(N \times N)$ unit matrix.

The factorization of the coefficients, along with (2), implies the "canonical factorization" of $\widehat{R}(\theta)$:

$$\begin{aligned}\widehat{R}(\theta) = {} & (f_+(-\theta)\,P_+ + f_-(-\theta)\,P_- + f_0(-\theta)\,P_0) \\ & \times((f_+(\theta))^{-1} P_+ + (f_-(\theta))^{-1} P_- + (f_0(\theta))^{-1} P_0).\end{aligned} \tag{4}$$

Like the projectors, the f's will also depend on q, the parameter of "q-deformation."

For $\widehat{R}(\theta)$ to be a braid matrix it must, by definition, satisfy the braid equation which provides matricial representation of the third Reidemeister move in the theory of classification of braids and knots. This means that defining (with "rapidities" (θ, θ'))

$$\widehat{R}(\theta) \otimes I = \widehat{R}_{12}(\theta),$$
$$I \otimes \widehat{R}(\theta) = \widehat{R}_{23}(\theta),$$

and

$$\begin{aligned}\widehat{B}_1 &\equiv \widehat{R}_{12}(\theta)\widehat{R}_{23}(\theta+\theta')\widehat{R}_{12}(\theta'), \\ \widehat{B}_2 &\equiv \widehat{R}_{23}(\theta')\widehat{R}_{12}(\theta+\theta')\widehat{R}_{23}(\theta'),\end{aligned} \tag{5}$$

the solutions for the coefficients must be found such that for a given set of explicitly defined $N^2 \times N^2$ dimensional projectors, one obtains

$$\widehat{B}_1 = \widehat{B}_2. \tag{6}$$

In Ref. 5, the coefficients for standard known solutions for $SO_q(N)$ and $Sp_q(N)$ were factorized as in (1) and new solutions were obtained, which were studied in subsequent papers.[15]

The *new classes* (denoted as $S\widehat{O}_q(N)$ and $S\widehat{p}_q(N)$, respectively) correspond to

$$f_+(\theta) = f_-(\theta) = 1 \tag{7}$$

and hence (with new solutions for f_0)

$$\begin{aligned}\widehat{R}(\theta) &= P_+ + P_- + \frac{f_0(-\theta)}{f_0(\theta)} P_0 \\ &= I \otimes I + (\frac{f_0(-\theta)}{f_0(\theta)} - 1) P_0 \qquad (8) \\ &\equiv I \otimes I + \omega(\theta) P_0. \qquad (9)\end{aligned}$$

In this paper, we analyze for the first time, the properties of P_0 that makes the solution of (9) possible. Later we consider $\omega(\theta)$ in that context. This turns out to be fruitful indeed.

III. $S\widehat{O}_q(N)$, $S\widehat{p}_q(N)$, AND TEMPERLEY-LIEB ALGEBRA

We define

$$(\omega(\theta), \omega(\theta'), \omega(\theta + \theta')) \equiv (\omega, \omega', \omega''). \tag{10}$$

Implementing (9) in (5), one obtains

$$\begin{aligned}(\widehat{B}_1 - \widehat{B}_2) = &(\omega + \omega' + \omega\omega' - \omega'')((P_0 \otimes I) - (I \otimes P_0)) \\ &+ \omega\omega'\omega''((P_0 \otimes I)(I \otimes P_0)(P_0 \otimes I) \\ &- (I \otimes P_0)(P_0 \otimes I)(I \otimes P_0)).\end{aligned} \tag{11}$$

We first define some notations:

(1) $\bar{i} \equiv N - i + 1$ (when $\bar{\bar{i}} = i$).
(2) (ij) as the $(N \times N)$ matrix with unity on row i and column j and all other elements zero.
(3) The q-brackets

$$[N \pm 1] = \frac{q^{N\pm1} - q^{-(N\pm1)}}{q - q^{-1}}. \tag{12}$$

We then have the projectors P_0 as $N^2 \times N^2$ matrices:[14]

(1) For $SO_q(N)$, $(N = 3, 4, 5, ...)$

$$([N-1]+1)P_0 = \sum_{i,j=1}^{N} q^{(\rho_{\bar{i}} - \rho_j)} (ij) \otimes (\overline{i}\,\overline{j}), \tag{13}$$

and

(2) For $Sp_q(N)$, $(N = 2, 4, 6, ...)$

$$([N+1]-1)P_0 = \sum_{i,j=1}^{N} q^{(\rho_{\bar{i}} - \rho_j)} (\epsilon_i \epsilon_j)((ij) \otimes (\overline{i}\,\overline{j})), \tag{14}$$

where

$$\begin{aligned}\epsilon_i &= 1, \quad (i \leq N/2),\\ \epsilon_i &= -1, \quad (i > N/2).\end{aligned} \tag{15}$$

Note that we restrict q to be real, positive throughout so that one obtains real P_0. Also, note that the parameters ρ are N-tuples:

1. For $SO(2n + 1)$:

$$\rho : (n - \frac{1}{2}, n - \frac{3}{2}, ..., \frac{1}{2}, 0, -\frac{1}{2}, ..., -n + \frac{1}{2}). \tag{16}$$

2. For $SO(2n)$:

$$\rho : (n - 1, n - 2, ..., 1, 0, 0, -1, ..., -n + 1). \tag{17}$$

3. For $Sp(2n)$:

$$\rho : (n, n - 1, ..., 1, -1, ..., -n). \tag{18}$$

The projectors, thus defined, can be shown to satisfy

$$(P_0 \otimes I)(I \otimes P_0)(P_0 \otimes I) = k^{-2}(P_0 \otimes I), \tag{19}$$

$$(I \otimes P_0)(P_0 \otimes I)(I \otimes P_0) = k^{-2}(I \otimes P_0), \tag{20}$$

where

$$k = ([N - 1] + 1), \quad \text{for } SO(N), \tag{21}$$

$$k = ([N + 1] - 1), \quad \text{for } Sp(N). \tag{22}$$

This is the core of the Temperley-Lieb algebra to be developed fully for spin chains, in Sec. IV. At this stage, implementing (19) and (20) in (11), one obtains

$$\begin{aligned}(\widehat{B}_1 - \widehat{B}_2) = (&\omega + \omega' + \omega\omega' - \omega''\\ &+k^{-2}\omega\omega'\omega'')((P_0 \otimes I) - (I \otimes P_0)).\end{aligned} \tag{23}$$

Hence the braid equation is satisfied, if

$$\omega + \omega' + \omega\omega' - \omega'' + k^{-2}\omega\omega'\omega'' = 0. \tag{24}$$

This nonlinear functional equation was solved in our previous paper.[15] Denoting the special cases as $S\widehat{O}$ and $S\widehat{p}$ henceforward, the solution is given by

$$\omega(\theta) = (\frac{\sinh(\eta - \theta)}{\sinh(\eta + \theta)} - 1), \tag{25}$$

where

$$(e^{\eta} + e^{-\eta}) = k = \frac{q^{N-\epsilon} - q^{-N+\epsilon}}{q - q^{-1}} + \epsilon \tag{26}$$

and

$$\begin{aligned}\epsilon &= 1 \quad \text{for } S\widehat{O}(N),\\ \epsilon &= -1 \quad \text{for } S\widehat{p}(N).\end{aligned}$$

We note the following points:

(1) Starting with the "standard solutions" of $\widehat{R}(\theta)$ (see (2.2) and (2.3) of Ref. 5) and letting $\theta \to \infty$, one obtains the results (2.8) of Ref. 5. From (2) of the present work, one obtains the result for $\widehat{R}(\theta)$ in terms of (P_+, P_0) or (P_-, P_0). Since P_0 has already been given, one may now obtain easily the explicit expressions for P_+ or P_-. They indeed correspond to (1.11) and (1.12) of Ref. 14. It may be easily verified that the coefficient of P_+ (or P_-) does not satisfy a constraint corresponding to (24). Thus, a Temperley-Lieb algebra is no longer obtained. Of the three projectors (P_+, P_-, P_0), only one projector, P_0, satisfies (19) and (20). Hence, P_0 is the crucial operator.

(2) An adequate solution of the nonlinear functional equation (24) in two variables (θ, θ') is not evident to start with. But it does exist and accordingly, an explicit manageable solution (given by (25) and (26)) is obtained.

(3) We can re-write (26) as

$$\cosh\eta = \frac{1}{2}k = \frac{1}{2}([N \mp 1] \pm 1) \tag{27}$$

(upper signs for $S\widehat{O}(N)$ and lower signs for $S\widehat{p}(N)$). We can also write

$$\sinh\eta = \pm\frac{1}{2}\sqrt{k^2 - 4} \tag{28}$$

for both cases $S\widehat{O}(N)$ and $S\widehat{p}(N)$, and hence $\sinh\eta$ can be chosen to be *positive or negative*, a point which we will revisit.

IV. SPIN CHAIN HAMILTONIAN AND TEMPERLEY-LIEB ALGEBRA

The chain Hamiltonian (and also higher order conserved quantities[5]) can be obtained directly as follows:

We define

$$\dot{\widehat{R}}(0) = (\frac{d}{d\theta}\widehat{R}(\theta))_{\theta=0}, \tag{29}$$

and with $(\dot{\widehat{R}}(0))_{l,l+1}$ acting on sites $(l, l+1)$, the Hamiltonian for an open chain of length r can be written as

$$H = \sum_{l=1}^{r-1} I \otimes \ldots \otimes I \otimes (\dot{\widehat{R}}(0))_{l,l+1} \otimes I \otimes \ldots \otimes I, \tag{30}$$

where I is the $N \times N$ unit matrix.

For a closed chain with circular boundary conditions, there is an additional term with

$$(\dot{\widehat{R}}(0))_{r,r+1}, \quad (r+1 \approx 1). \tag{31}$$

For our case

$$\dot{\widehat{R}}(0) = -(2\coth\eta)P_0 = \mp(\frac{2k}{(k^2-4)^{1/2}})P_0 \tag{32}$$

for the upper and lower signs in (28), respectively.

Hence for open r-chains, we have

$$H = \mp(\frac{2k}{(k^2-4)^{1/2}})(\sum_{l=1}^{r-1} I \otimes \ldots \otimes I \otimes (P_0)_{l,l+1} \otimes I \otimes \ldots \otimes I). \tag{33}$$

We define

$$X_l \equiv I \otimes \ldots \otimes I \otimes (P_0)_{l,l+1} \otimes I \otimes \ldots \otimes I. \tag{34}$$

Then

$$H = -(2\coth\eta)(\sum_{l=1}^{r-1} X_l), \tag{35}$$

where η is non-zero, and positive or negative according to the sign chosen in (28).

From (19)–(22) with k given by (27),

$$\begin{aligned} X_l X_{l+1} X_l &= k^{-2} X_l, \\ X_l^2 &= X_l, \\ X_l X_m &= X_m X_l \quad (|l-m| > 1). \end{aligned} \tag{36}$$

Thus, the chain Hamiltonian is obtained as a sum over generators of the Temperley-Lieb algebra, defined by (36).

Defining

$$X'_l = kX_l = (e^{\eta} + e^{-\eta})X'_l, \tag{37}$$

one obtains

$$\begin{aligned} X'_l X'_{l+1} X'_l &= X'_l, \\ X'^2_l &= (e^{\eta} + e^{-\eta})X'_l, \\ X'_l X'_m &= X'_m X'_l \quad (|l-m| > 1), \end{aligned} \tag{38}$$

a standard form of Temperley-Lieb algebra.

We note the following points:

1. As we mentioned before, the link between spin chains and Temperley-Lieb algebras is a well-studied subject.[6] But here we have more than the defining relations (36) or (38). We have, for all N, explicit $N^2 \times N^2$ matrix realizations of the generators: (implementing (13)–(18) in (34) and (37)). This, as will be displayed below, enables one to construct eigenstates and eigenvalues of chain Hamiltonians for all N.
2. The two signs in (28) will be seen to correspond to inversion of the sign of eigenvalues of H. They correspond to two different regimes.

V. EIGENSTATES AND EIGENVALUES OF P_0 AND ACTION OF H

We start by presenting the action of the projector P_0 on product states and then derive that of the Hamiltonian (33). The definitions (12), (13), and the explicit particular cases of Appendix A imply that the action of P_0 on general mixed states selects out specific linear combinations of states

$$\left|i\bar{i}\right\rangle = |i\rangle \otimes |N-i+1\rangle\,.$$

In terms of

$$P'_0 = ([N-\epsilon] + \epsilon)P_0, \tag{39}$$

one obtains from (13)–(18), for $S\widehat{O}(N)$:

$$P'_0(\sum_{a=1}^{N} x_a\,|a\rangle) \otimes (\sum_{b=1}^{N} y_b\,|b\rangle) = \sum_{i=1}^{N} q^{(\rho_{\bar{i}}-\rho_i)} x_i y_{\bar{i}} \left|i\bar{i}\right\rangle, \tag{40}$$

and for $S\widehat{p}(N)$:

$$P'_0(\sum_{a=1}^{N} x_a\,|a\rangle) \otimes (\sum_{b=1}^{N} y_b\,|b\rangle) = \sum_{i=1}^{N} q^{(\rho_{\bar{i}}-\rho_i)} (\epsilon_i x_i)(\epsilon_{\bar{i}} y_{\bar{i}}) \left|i\bar{i}\right\rangle, \tag{41}$$

with the ϵ's defined in (15).

Explicit results of Appendix A can now be implemented as follows.

A. $S\widehat{O}(3)$

$$\begin{aligned}&P_0'(x_1\,|1\rangle + x_2\,|2\rangle + x_{\bar{1}}\,|\bar{1}\rangle)\\&\otimes(y_1\,|1\rangle + y_2\,|2\rangle + y_{\bar{1}}\,|\bar{1}\rangle)\\&= (q^{-1/2}x_1y_{\bar{1}} + x_2y_2 + q^{1/2}x_{\bar{1}}y_1)\,|\Psi\rangle\,,\end{aligned} \tag{42}$$

where

$$|\Psi\rangle \equiv (q^{-1/2}\,|1\bar{1}\rangle + |22\rangle + q^{1/2}\,|\bar{1}1\rangle). \tag{43}$$

The states $(|1\rangle\,, |2\rangle\,, |\bar{1}\rangle)$ may be taken to correspond to spin projections $(1, 0, -1)$. This $|\Psi\rangle$ turns out to be an eigenstate of P_0 with unity for eigenvalue:

$$P_0\,|\Psi\rangle = |\Psi\rangle \quad (\text{from } P_0'\,|\Psi\rangle = (q^{-1} + 1 + q)\,|\Psi\rangle\,), \tag{44}$$

(using (39) with $N = 3$, $([N-1]+1) = (q^{-1}+1+q)$). From (42)

$$P_0'(|1\bar{1}\rangle\,, |22\rangle\,, |\bar{1}1\rangle) = (q^{-1/2}, 1, q^{1/2})\,|\Psi\rangle \tag{45}$$

and

$$P_0'\,|ij\rangle = 0 \quad (j \neq \bar{i}). \tag{46}$$

Strictly analogous results hold for all $S\widehat{O}(N)$ and $S\widehat{p}(N)$. This is already pointed out for the cases of Appendix A.

B. $S\widehat{O}(4)$

$$P_0' = (q^{-2} + 2 + q^2)P_0.$$

Then

$$\begin{aligned}&P_0'(x_1\,|1\rangle + x_2\,|2\rangle + x_{\bar{2}}\,|\bar{2}\rangle + x_{\bar{1}}\,|\bar{1}\rangle)\\&\otimes(y_1\,|1\rangle + y_2\,|2\rangle + y_{\bar{2}}\,|\bar{2}\rangle + y_{\bar{1}}\,|\bar{1}\rangle)\\&= (q^{-1}x_1y_{\bar{1}} + x_2y_{\bar{2}} + x_{\bar{2}}y_2 + qx_{\bar{1}}y_1)\,|\Psi\rangle\,,\end{aligned} \tag{47}$$

where

$$|\Psi\rangle \equiv q^{-1}\,|1\bar{1}\rangle + |2\bar{2}\rangle + |\bar{2}2\rangle + q\,|\bar{1}1\rangle \tag{48}$$

and

$$P_0\,|\Psi\rangle = |\Psi\rangle\,. \tag{49}$$

As in (45) the non-zero results are obtained for only

$$P_0'(|1\bar{1}\rangle, |2\bar{2}\rangle, |\bar{2}2\rangle, |\bar{1}1\rangle) = (q^{-1}, 1, 1, q)\,|\Psi\rangle\,. \tag{50}$$

The states $(|1\rangle\,, |2\rangle\,, |\bar{2}\rangle, |\bar{1}\rangle)$ correspond to spin projections $(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})$. Our notation here generalizes smoothly to any N.

C. $S\widehat{p}(4)$

$$P_0' = (q^{-4} + q^{-2} + q^2 + q^4)P_0. \tag{51}$$

Then

$$P_0'(x_1\,|1\rangle + x_2\,|2\rangle + x_{\bar{2}}\,|\bar{2}\rangle + x_{\bar{1}}\,|\bar{1}\rangle)$$
$$\otimes(y_1\,|1\rangle + y_2\,|2\rangle + y_{\bar{2}}\,|\bar{2}\rangle + y_{\bar{1}}\,|\bar{1}\rangle)$$
$$= (q^{-2}x_1 y_{\bar{1}} + q^{-1}x_2 y_{\bar{2}} - q x_{\bar{2}} y_2 - q^2 x_{\bar{1}} y_1)\,|\Psi\rangle\,, \tag{52}$$

where

$$|\Psi\rangle \equiv q^{-2}\,|1\bar{1}\rangle + q^{-1}\,|2\bar{2}\rangle - q\,|\bar{2}2\rangle - q^2\,|\bar{1}1\rangle \tag{53}$$

and

$$P_0\,|\Psi\rangle = |\Psi\rangle\,. \tag{54}$$

Non-zero actions:

$$P_0'(|1\bar{1}\rangle, |2\bar{2}\rangle, |\bar{2}2\rangle, |\bar{1}1\rangle) = (q^{-2}, q^{-1}, -q, -q^2)\,|\Psi\rangle\,. \tag{55}$$

Let us now consider the action of H on an open chain of length r.

Similar to (39), $P_0' = kP_0$, we now define

$$X_l' = kX_l = \sum_{l=1}^{r-1} I\otimes ... \otimes I \otimes (P_0')_{l,l+1} \otimes I \otimes ... \otimes I \tag{56}$$

with ($l = 1, ..., r - 1$), and

$$k = ([N \mp 1] \pm 1)$$
$$= \frac{q^{(N\mp 1)} - q^{-(N\mp 1)}}{q - q^{-1}} \pm 1,$$

for $S\widehat{O}(N)$ and $S\widehat{p}(N)$, respectively, where we consider real values of q.

Using (28), we can now write

$$H = \mp\frac{2k}{\sqrt{k^2-4}}\sum_{l=1}^{r-1} X_l' = -\frac{1}{\sinh\eta}\sum_{l=1}^{r-1} X_l' \tag{57}$$

$$\equiv -(\sinh\eta)^{-1} H'. \tag{58}$$

Now

$$H'\,|i_1 i_2 ... i_r\rangle = \sum_{l=1}^{r-1} |i_1 i_2 ... i_{l-1}\rangle\,((P_0')\,|i_l i_{l+1}\rangle)\,|i_{l+2} ... i_r\rangle \tag{59}$$

and

$$P_0'\,|i_l i_{l+1}\rangle = \delta(\bar{i}_l, i_{l+1}) q^{(\rho_{\bar{i}} - \rho_i)}\epsilon_i \epsilon_{\bar{i}}\,|i_l \bar{i}_l\rangle \tag{60}$$

with the n-tuples defined by (16)–(18), and with ϵ's as indicated by (15) for $S\widehat{p}(N)$ (each $\epsilon = 1$ for $S\widehat{O}(N)$).

Let us concentrate on the explicit case of $S\widehat{O}(3)$. Consider, for example, the $S\widehat{O}(3)$ 4-chain with mixed product states

$$|X\rangle \equiv (a_1\,|1\rangle + a_2\,|2\rangle + a_{\bar{1}}\,|\bar{1}\rangle)$$
$$\otimes(b_1\,|1\rangle + b_2\,|2\rangle + b_{\bar{1}}\,|\bar{1}\rangle)$$
$$\otimes(c_1\,|1\rangle + c_2\,|2\rangle + c_{\bar{1}}\,|\bar{1}\rangle)$$
$$\otimes(d_1\,|1\rangle + d_2\,|2\rangle + d_{\bar{1}}\,|\bar{1}\rangle)$$
$$\equiv |x\rangle_1 \otimes |x\rangle_2 \otimes |x\rangle_3 \otimes |x\rangle_4\,. \tag{61}$$

Defining, as in (43)

$$|\Psi\rangle \equiv q^{-1/2}\left|1\bar{1}\right\rangle + |22\rangle + q^{1/2}\left|\bar{1}1\right\rangle, \tag{62}$$

we now have

$$\begin{aligned} H_4'\,|x\rangle = {} & (a_1 b_{\bar{1}} q^{-1/2} + a_2 b_2 + a_{\bar{1}} b_1 q^{1/2})\,|\Psi\rangle\,|x\rangle_3\,|x\rangle_4 \\ & + (b_1 c_{\bar{1}} q^{-1/2} + b_2 c_2 + b_{\bar{1}} c_1 q^{1/2})\,|x\rangle_1\,|\Psi\rangle\,|x\rangle_4 \\ & + (c_1 d_{\bar{1}} q^{-1/2} + c_2 d_2 + c_{\bar{1}} d_1 q^{1/2})\,|x\rangle_1\,|x\rangle_2\,|\Psi\rangle\,. \end{aligned} \tag{63}$$

A generalization to a chain of length r is quite straight forward. In notations that are evident

$$H'(|x\rangle_1 \ldots |x\rangle_r) = \sum_{l=1}^{r-1} f_l\,|x\rangle_1 \ldots |x\rangle_{l-1}\,|\Psi\rangle\,|x\rangle_{l+2} \ldots |x\rangle_r\,, \tag{64}$$

where

$$f_l = (a_1^{(l)} b_{\bar{1}}^{(l+1)} q^{-1/2} + a_2^{(l)} b_2^{(l+1)} + a_{\bar{1}}^{(l)} b_1^{(l+1)} q^{1/2}). \tag{65}$$

For $S\widehat{O}(4)$, $S\widehat{p}(4)$, and so on, one can easily implement (58) and (59), with previous definitions.

VI. TIME EVOLUTION OF SPIN CHAINS AND DATA TRANSMISSION

A. Evolution in time

We are now in a position to start studying the evolution in time t of a chain under the action of the operator e^{-iHt}. As is often the case, we try to display some basic features by presenting results explicitly for a few restricted simple cases and indicating how to generalize them.

Consider an $S\widehat{O}(3)$ chain of spins, the projections for spin 1 being denoted as

$$(|+\rangle\,,\,|0\rangle\,,\,|-\rangle) \equiv (|1\rangle\,,\,|0\rangle\,,\,\left|\bar{1}\right\rangle). \tag{66}$$

From (42)–(46) one sees then the iterative actions of H on the building blocks

$$|\Psi\rangle\,|i\rangle\,,\,|i\rangle\,|\Psi\rangle\,,\,|i\rangle\,|\Psi\rangle\,|j\rangle\,,$$

where

$$|\Psi\rangle = q^{-1/2}\left|1\bar{1}\right\rangle + |22\rangle + q^{1/2}\left|\bar{1}1\right\rangle \tag{67}$$

and (i, j) take the values $(1, 0, \bar{1})$, are the essential ingredients, along with the basic initial results:

$$\begin{aligned} P_0'(\left|1\bar{1}\right\rangle, |22\rangle\,, \left|\bar{1}1\right\rangle) &= (q^{-1} + 1 + q)P_0(\left|1\bar{1}\right\rangle, |22\rangle\,, \left|\bar{1}1\right\rangle) \\ &= (q^{-1/2}, 1, q^{1/2})(q^{-1/2}\left|1\bar{1}\right\rangle \\ &\quad + |22\rangle + q^{1/2}\left|\bar{1}1\right\rangle) \\ &\equiv (q^{-1/2}, 1, q^{1/2})\,|\Psi\rangle\,, \\ P_0'\,|ij\rangle &= 0 \quad (j \neq \bar{i}), \end{aligned}$$

and

$$k P_0\,|\Psi\rangle \equiv P_0'\,|\Psi\rangle = (q^{-1} + 1 + q)\,|\Psi\rangle \equiv k\,|\Psi\rangle\,. \tag{68}$$

From (28), (32)–(35), and (68), for an open r-chain

$$H = \lambda \sum_{l=1}^{r-1} I \otimes \ldots \otimes I \otimes (P_0')_{l,l+1} \otimes I \otimes \ldots \otimes I, \tag{69}$$

where (corresponding to the sign of η chosen in (28))

$$\lambda \equiv \mp \frac{2}{\sqrt{k^2-4}}, \tag{70}$$

which (from (68)) is real for q real, positive (which we assume to be the case). Note that for a closed chain the summation (69) would include an extra term $l = r$ with $(r + 1) \approx 1$.

Defining

$$H \equiv \lambda H', \tag{71}$$

we consider the series expansion (with $I \otimes I = I_9$ for $S\widehat{O}(3)$):

$$\begin{aligned} e^{-iHt} &= e^{-i\lambda t H'} \\ &= I_9 + (-i\lambda t)H' + \frac{1}{2!}(-i\lambda t)^2 (H')^2 \\ &\quad + \frac{1}{3!}(-i\lambda t)^3 (H')^3 + \ldots \end{aligned} \tag{72}$$

up to any chosen order in t.

Suppose that the spin states

$$|\Psi\rangle\,|i\rangle\,,\ |i\rangle\,|\Psi\rangle\,,\ |i\rangle\,|\Psi\rangle\,|j\rangle$$

correspond, respectively, to the sites

$$(l, l+1, l+2), \quad (l-1, l, l+1), \quad (l-1, l, l+1, l+2).$$

Defining

$$\begin{aligned} H'_{(3)} &= P_0' \otimes I + I \otimes P_0', \\ H'_{(4)} &= P_0' \otimes I \otimes I + I \otimes P_0' \otimes I + I \otimes I \otimes P_0', \end{aligned}$$

they will always be implicitly assumed to correspond to the appropriate sites as the relevant parts of the total H' acting on the total chain. Thus, $H'_{(3)}\,|\Psi\rangle\,|i\rangle$ corresponds to the terms of H' acting on the sites $(l, l + 1, l + 2)$ and so on.

One obtains from (42)–(46),

$$H'_{(3)}\,|\Psi\rangle\,|i\rangle = k\,|\Psi\rangle\,|i\rangle + |i\rangle\,|\Psi\rangle\,, \tag{73}$$

$$H'_{(3)}\,|i\rangle\,|\Psi\rangle = k\,|i\rangle\,|\Psi\rangle + |\Psi\rangle\,|i\rangle\,. \tag{74}$$

Iterating, one obtains (see Appendix B)

$$(H'_{(3)})^p(|\Psi\rangle\,|i\rangle) = A_p\,|\Psi\rangle\,|i\rangle + B_p\,|i\rangle\,|\Psi\rangle\,, \tag{75}$$

$$(H'_{(3)})^p\,|i\rangle\,|\Psi\rangle = A_p\,|i\rangle\,|\Psi\rangle + B_p\,|\Psi\rangle\,|i\rangle\,, \tag{76}$$

where

$$A_p = \frac{1}{2}((k+1)^p + (k-1)^p), \tag{77}$$

$$B_p = \frac{1}{2}((k+1)^p - (k-1)^p). \tag{78}$$

Next assuming $j \neq \bar{i}$, we have

$$H'_{(4)}(|i\rangle\,|\Psi\rangle\,|j\rangle) = |\Psi\rangle\,|ij\rangle + k\,|i\rangle\,|\Psi\rangle\,|j\rangle + |ij\rangle\,|\Psi\rangle \tag{79}$$

or

$$(H'_{(4)} - k)(|i\rangle\,|\Psi\rangle\,|j\rangle) = |ij\rangle\,|\Psi\rangle + |\Psi\rangle\,|ij\rangle\,. \tag{80}$$

Now, from (73), (74), and (80)

$$(H'_{(4)} - k)^2(|i\rangle\,|\Psi\rangle\,|j\rangle) = 2\,|i\rangle\,|\Psi\rangle\,|j\rangle \tag{81}$$

since $P'_0\,|ij\rangle = 0, (j \neq \bar{i})$. Hence

$$(H'_{(4)} - k)^{2n}(|i\rangle\,|\Psi\rangle\,|j\rangle) = 2^{(n-1)}(|i\rangle\,|\Psi\rangle\,|j\rangle), \tag{82}$$

$$(H'_{(4)} - k)^{2n+1}(|i\rangle\,|\Psi\rangle\,|j\rangle) = 2^{(n-1)}(|ij\rangle\,|\Psi\rangle + |\Psi\rangle\,|ij\rangle\,. \tag{83}$$

For $|j\rangle = \left|\bar{i}\right\rangle$ there are additional terms. One obtains

$$H'_{(4)}(|i\rangle\,|\Psi\rangle\,\left|\bar{i}\right\rangle) = |\Psi\rangle\,\left|i\bar{i}\right\rangle + k\,|i\rangle\,|\Psi\rangle\,\left|\bar{i}\right\rangle + \left|i\bar{i}\right\rangle|\Psi\rangle\,, \tag{84}$$

$$H'_{(4)}(|\Psi\rangle\,\left|i\bar{i}\right\rangle) = k\,|\Psi\rangle\,\left|i\bar{i}\right\rangle + |i\rangle\,|\Psi\rangle\,\left|\bar{i}\right\rangle + q^{\delta_i}\,|\Psi\rangle\,|\Psi\rangle\,, \tag{85}$$

$$H'_{(4)}(\left|i\bar{i}\right\rangle|\Psi\rangle) = k\,\left|i\bar{i}\right\rangle|\Psi\rangle + |i\rangle\,|\Psi\rangle\,\left|\bar{i}\right\rangle + q^{\delta_i}\,|\Psi\rangle\,|\Psi\rangle\,, \tag{86}$$

where $\delta_i = (-\frac{1}{2}, 0, \frac{1}{2})$, respectively, for

$$i = (1, 2, \bar{1}). \tag{87}$$

Hence,

$$\begin{aligned}(H'_{(4)})^2(|i\rangle\,|\Psi\rangle\,\left|\bar{i}\right\rangle) = (k^2+2)\,|i\rangle\,|\Psi\rangle\,\left|\bar{i}\right\rangle \\ +2k(|\Psi\rangle\,\left|i\bar{i}\right\rangle + \left|i\bar{i}\right\rangle|\Psi\rangle) \\ +2q^{\delta_i}\,|\Psi\rangle\,|\Psi\rangle\end{aligned} \tag{88}$$

and again

$$\begin{aligned}H'_{(4)}(|\Psi\rangle\,|\Psi\rangle) &= 2k\,|\Psi\rangle\,|\Psi\rangle + \sum_i q^{\delta_i}\,|i\rangle\,|\Psi\rangle\,\left|\bar{i}\right\rangle \\ &= 2k\,|\Psi\rangle\,|\Psi\rangle + (q^{-1/2}\,|1\rangle\,|\Psi\rangle\,\left|\bar{1}\right\rangle \\ &\quad + |2\rangle\,|\Psi\rangle\,|2\rangle + q^{1/2}\,\left|\bar{1}\right\rangle|\Psi\rangle\,|1\rangle).\end{aligned} \tag{89}$$

Using the set (84)–(89) one can now iterate. The way to proceed and the essential ingredients have been all presented above. We will not write down the general result for $(H'_{(4)})^n(|i\rangle\,|\Psi\rangle\,\left|\bar{i}\right\rangle)$. In all the examples above there is one feature in common: The eigenstates $|\Psi\rangle$ of P'_0 appear under the action of H' and move along the chain under iterations. They move both forward and backward. There can be multiple $|\Psi\rangle$ depending on the initial state. The iterations above are to be implemented in (72). Using systematically the results above one can start to study the evolution of an initial chain configuration.

B. Transmission of data along a chain

For clarity and relative simplicity we start with an open 6-chain of $S\widehat{O}(3)$ spin states (66). The initial configuration is assumed to be (at $t = 0$)

$$|X\rangle_{(0)} = (c_1 \left|\bar{1}1\right\rangle + c_2 \left|1\bar{1}\right\rangle) \left|1111\right\rangle \tag{90}$$

$$\equiv c_1 \left|X\right\rangle_1 + c_2 \left|X\right\rangle_2 . \tag{91}$$

(We do not immediately normalize $|X\rangle$ for convenient generalization to more parameters, such as, that to start with (99), considered later.) As will be shown below, time evolution under the action of e^{-iHt} will generate (at time t) a mutually orthogonal set of states including

$$\left|1111\right\rangle (d_1 \left|\bar{1}1\right\rangle + d_2 \left|1\bar{1}\right\rangle). \tag{92}$$

The other states at a finite non-zero t will be (apart from (90)) sequences

$$\begin{aligned}&(\left|X\right\rangle_1 , \left|X\right\rangle_2 , \left|\Psi\right\rangle \left|1111\right\rangle , \left|1\right\rangle \left|\Psi\right\rangle \left|111\right\rangle ,\\ &\left|11\right\rangle \left|\Psi\right\rangle \left|11\right\rangle , \left|111\bar{1}11\right\rangle , \left|111221\right\rangle , \left|111122\right\rangle),\end{aligned} \tag{93}$$

whose coefficients can be obtained (see Appendix C).

With t increasing, the coefficients of the above set, (i.e., (92) and (93)) will continue to change. Otherwise, the set will be stable, no new basis states of the 6-chain will appear. This is a consequence of the specific properties of our H. It will be shown that it is sufficient to implement the series development

$$\begin{aligned}e^{-iHt} &= e^{-i(\lambda t)H'}\\ &= I_9 - i(\lambda t)H' - \frac{1}{2!}(\lambda t)^2(H')^2 + \frac{i}{3!}(\lambda t)^3(H')^3\\ &\quad + \frac{1}{4!}(\lambda t)^4(H')^4 - \frac{i}{5!}(\lambda t)^5(H')^5 + O(t^6).\end{aligned} \tag{94}$$

Evaluating finally,

$$(H')^p(c_1 \left|X\right\rangle_1 + c_2 \left|X\right\rangle_2)$$

for $p = (0, 1, 2, 3, 4, 5)$ one already obtains states of the type (92) (along with others orthogonal to it as given in (93)). Moreover, (d_1, d_2) is obtained explicitly in terms of (c_1, c_2, λ, t) where t is given (for any chosen origin) and from (70) (restricting the values of q for definiteness, to $q = 1 + \delta$, $\delta > 0$)

$$\lambda = \mp\frac{2}{\sqrt{(3)^2 - 4}} = \mp\frac{2}{\sqrt{5}} \tag{95}$$

for $S\widehat{O}(3)$ (i.e., for $k = 3$). The sign ambiguity in (70) corresponds to the two possible determinations of η (as explained in (28)) corresponding to two possible regimes.

Next, one can easily invert the relations and thus extract (c_1, c_2) from (d_1, d_2, λ, t). Thus, the initial mixed state at the left of the chain can be extracted by precise observation of the specific mixed state (92) at the right end of the chain at a finite time t.

In this precise sense, we say that the initial state $(c_1 \left|\bar{1}1\right\rangle + c_2 \left|1\bar{1}\right\rangle)$ at the left has been transmitted to the right as $(d_1 \left|\bar{1}1\right\rangle + d_2 \left|1\bar{1}\right\rangle)$ where (c_1, c_2) can be recovered from (d_1, d_2).

From the results of Appendix C one obtains

$$d_1 = c_1x_1 + c_2x_2, \tag{96}$$

$$d_2 = c_1y_1 + c_2y_2, \tag{97}$$

where (x_1, y_1), (x_2, y_2) are given in the appendix. From (C19) and (C21) one sees (since λ and k are known) that the coefficient of (t^3) in x_2 gives directly c_2 from d_1. One then easily extracts c_1 also from the coefficients of powers of t in (d_1, d_2). Thus, our goal is attained.

Apart from the development (72) in powers of t, we can also set

$$q = 1 + \delta \tag{98}$$

and assuming δ to be small, use a series development in powers of δ to extract information more readily concerning the initial state from that at time t. Let us illustrate this, very briefly, using a simple example.

Generalizing (90) to

$$|x\rangle_{(0)} = (a\,|\overline{1}1\rangle + b\,|22\rangle + c\,|1\overline{1}\rangle)\,|1111\rangle\,, \tag{99}$$

$$\begin{aligned} H'\,|x\rangle_{(0)} = (q^{-1/2}a + b + q^{1/2}c)\,|\Psi\rangle\,|1111\rangle \\ +cq^{1/2}\,|1\rangle\,|\Psi\rangle\,|111\rangle \end{aligned} \tag{100}$$

$$\begin{aligned} = ((a + c) - \frac{1}{2}\delta(a - c) + b)\,|\Psi\rangle\,|1111\rangle \\ +c(1 + \frac{1}{2}\delta)\,|1\rangle\,|\Psi\rangle\,|111\rangle + O(\delta^2). \end{aligned} \tag{101}$$

Thus, in the corresponding generalizations of (96) and (97) one can separate $(a + c)$ and $(a - c)$ and hence (a, c) by extracting coefficients of powers of δ.

This can be more helpful for more elaborate initial states. A smaller number of powers of t will be needed to extract the initial parameters from the generalizations of (d_1, d_2) above.

The special significance of the point $q = 1$ will be emphasized at the end of Sec. VII in the context of entanglement entropy. Here we note that a supplementary series development about $q = 1$ can help in another context. The dependence of our model on the quantum deformation parameter q is indeed a central feature.

One may compare our results above with the study of "Quantum communication through an unmodulated spin chain" in Ref. 7. There one has only Pauli matrices and only two possible spin states. But the Hamiltonian couples all possible pair of sites and static magnetic fields are present. The action of e^{-iHt} is studied numerically. The specific structure of our Hamiltonian (not only as here, for $N = 3$ but also for $N > 3$ through straightforward generalization) make explicit computations feasible.

To illustrate the above statements we consider, for $S\widehat{O}(3)$, a particularly simple initial configuration.

Suppose the chain C is given symbolically (with, $1, 2, \overline{1}$, corresponding to spin projections $(+1, 0, -1)$, respectively) by

$$C_0 = (...1111_{(p)}\overline{1}_{(p+1)}\overline{111}...) \tag{102}$$

with all sites up to, say p in state 1 and then all sites in state $\overline{1}$.

Gathering together the definitions and notations (56)–(59) in the compact notation

$$\begin{aligned} (-iHt) = \zeta[(P_{0(12)} \otimes I \otimes I \otimes ...) + (I \otimes P_{0(23)} \otimes I \otimes ...) \\ +(I \otimes I \otimes P_{0(34)} \otimes ...) + ...] \end{aligned} \tag{103}$$

$$\equiv \sum_{p=0} \zeta(H_{p,p+1}). \tag{104}$$

To start with, only $H_{p,\,p\,+\,1}$ will have a non-zero action on C.

$$HC_0 = ...111(\Psi_{p,p+1})\overline{111}..., \tag{105}$$

where $\Psi_{p,\,p+1}$ (given by (43)) is

$$(q^{-1/2}\,|1\overline{1}\rangle + |22\rangle + q^{1/2}\,|\overline{1}1\rangle). \tag{106}$$

In the action of e^{-iHt}, the Ψ states then spread out as follows:

$$\begin{aligned}
&(-iHt)^2 \quad : \\
&H\Psi_{p,p+1} \to (\Psi_{p-1,p}, \Psi_{p,p+1}, \Psi_{p+1,p+2}),
\end{aligned} \tag{107}$$

$$\begin{aligned}
&(-iHt)^3 \quad : \\
H\Psi_{p,p} &\to (\Psi_{p-2,p-1}, \Psi_{p-1,p}, \Psi_{p,p+1}), \\
H\Psi_{p,p+1} &\to (\Psi_{p-1,p}, \Psi_{p,p+1}, \Psi_{p+1,p+2}), \\
H\Psi_{p+1,p+2} &\to (\Psi_{p,p+1}, \Psi_{p+1,p+2}, \Psi_{p+2,p+3}).
\end{aligned} \tag{108}$$

One already sees, schematically, how the Ψ states are generated, move forward and backward, crossover and acquire coefficients corresponding to different terms

$$\frac{1}{n!}(-iHt)^n \quad (n = 1, 2, 3).$$

Already the multiplicities counting the contributions from different n's (with appropriate coefficients for each order) are for

$$\begin{aligned}
&(\Psi_{p-2,p-1}, \Psi_{p-1,p}, \Psi_{p,p+1}, \Psi_{p+1,p+2}, \Psi_{p+2,p+3}) \\
&\to (1, 3, 5, 3, 1).
\end{aligned} \tag{109}$$

After r steps (i.e., up to order t^r), the above sequence is

$$(..., 2r-5, 2r-3, 2r-1, 2r-3, 2r-5, ...).$$

An initial state less simple can give a much more complex pattern. For initial C

$$\begin{aligned}
C_{+-} &: (1\bar{1}1\bar{1}1 ... \bar{1}1\bar{1}), \\
C_{-+} &: (\bar{1}1\bar{1}1\bar{1} ... 1\bar{1}1), \\
C_{00} &: (22222...222),
\end{aligned}$$

even the term $(-iHt)$ in the expansion of e^{-iHt} generates Ψ states for each pair of sites $(p, p+1)$.

These states Ψ are the basic building blocks of our formalism. They will be seen to be *entangled* states, and their (q, N)-dependent entropy will be studied in Sec. VII.

VII. (q, N)-DEPENDENT ENTANGLEMENT ENTROPY OF EIGENSTATES OF P_0

Acting on the pure product states $|i\bar{i}\rangle$, $(\bar{i} = N - i + 1)$ for both $S\widehat{O}(N)$ and $S\widehat{p}(N)$ the projector P_0 creates its eigenstate

$$P_0 \, |i\bar{i}\rangle = \sum_{j=1}^{N} ((P_0)_{(ji),(\overline{ji})}) \, |j\bar{j}\rangle \approx |\Psi\rangle \,, \tag{110}$$

the matrix elements of P_0 in (110), being defined as in (12)–(18).

Does P_0 thus generate entanglement? We give an affirmative answer below, evaluate the entanglement entropy to quantify it and analyze the (q, N)-dependence, q being the parameter of quantum deformation. We formulate the q-dependence, separately for different values of N.

A. $S\widehat{O}(3)$

As usual, we start with $S\widehat{O}(3)$ (see (42)–(46)) and (A1)–(A4)) and as essential step, normalize $|\Psi\rangle$ as below, denoting it now by $|\Psi\rangle_{(n)}$. Define

$$|\Psi\rangle_{(n)} = (q^{-1}+1+q)^{-1/2}(q^{-1/2}\,|{+}{-}\rangle + |00\rangle + q^{1/2}\,|{-}{+}\rangle)$$

$$\equiv (q^{-1}+1+q)^{-1/2}(q^{-1/2}\,\left|1\bar{1}\right\rangle + |22\rangle + q^{1/2}\,\left|\bar{1}1\right\rangle). \tag{111}$$

That this is an entanglement state, is evident immediately. Attempting to re-express it as a product state

$$(c_+\,|+\rangle + c_0\,|0\rangle + c_-\,|-\rangle)(d_+\,|+\rangle + d_0\,|0\rangle + d_-\,|-\rangle)$$

one runs directly into contradictory constraints on (c_i, d_j) coefficients.

To quantify the entanglement one first notes that the eigenstate $|\Psi\rangle_{(n)}$ satisfying

$$P_0\,|\Psi\rangle_{(n)} = |\Psi\rangle_{(n)} \tag{112}$$

is already Schmidt decomposed as

$$|\Psi\rangle_{(n)} = \sum_i a_{i\bar{i}}\,\left|i\bar{i}\right\rangle. \tag{113}$$

Hence, without passing via the density matrix (and without using a $\log 2$ basis) one obtains the von-Neumann entropy[16,17] as

$$S = -\sum_i |a_{i\bar{i}}|^2 \ln |a_{i\bar{i}}|^2, \quad i = (+,0,-). \tag{114}$$

From (111), (113), and (114), one obtains

$$S(q) = S(q^{-1}) \tag{115}$$

$$= \ln(q^{-1}+1+q) - \frac{(q-q^{-1})}{(q^{-1}+1+q)}\ln q.$$

From $q = 1$ one obtains the maximum entropy as

$$S(1) = S(\max) = \ln 3. \tag{116}$$

To first order in $\epsilon > 0$,

$$S(1 \mp \epsilon) = \ln 3 \pm \frac{2}{3}\epsilon \ln(1 \mp \epsilon) < S(1). \tag{117}$$

Consistently with (115) (namely, $S(q) = S(q^{-1})$)

$$S(q) \to 0 \text{ as } q \to \infty \text{ or } q \to 0. \tag{118}$$

After displaying the q-dependence for $N = 3$ we explore below also the N-dependence. As a first step we move up from $N = 3$ to $N = 4$.

B. $S\widehat{O}(4)$

From (47)–(50) and (A6)–(A8) we define now the normalized eigenstate of P_0 as

$$|\Psi\rangle_{(n)} = (q^{-2}+2+q^2)^{-1/2}(q^{-1}\,\left|1\bar{1}\right\rangle + \left|2\bar{2}\right\rangle + \left|\bar{2}2\right\rangle + q\,\left|\bar{1}1\right\rangle). \tag{119}$$

The corresponding entropy is obtained as

$$S(q) = S(q^{-1}) = 2\ln(q^{-1}+q) - 2\frac{(q-q^{-1})}{(q+q^{-1})}\ln q. \tag{120}$$

Now, again for $q = 1$,

$$S(\max) = S(1) = \ln 4, \tag{121}$$

$$S(1 \mp \epsilon) = \ln 4 \pm 4\epsilon \ln(1 \mp \epsilon) < S(1). \tag{122}$$

Once again (118)

$$S(q) \to 0 \text{ as } q \to \infty \text{ or } q \to 0. \tag{123}$$

Hence as N increases from 3 to 4, $S(\max)$ moves up from $\ln 3$ to $\ln 4$ and falls a bit more steeply, but again symmetrically in (q, q^{-1}) to vanishing asymptotic values as $q \to \infty$ and $q \to 0$.

C. $S\widehat{p}(4)$

For $N = (4, 6, 8, ...)$, i.e., for each such even N, one has the projector P_0 for $S\widehat{p}(N)$ as well as for $S\widehat{O}(N)$. Though this paper is mostly devoted to a detailed study of $S\widehat{O}(3)$, after showing how the entropy depends on N for $S\widehat{O}(N)$ by presenting the results for $S\widehat{O}(4)$, we also present briefly the results for $S\widehat{p}(4)$ to display both the analogies and the differences in this respect between $S\widehat{O}(4)$ and $S\widehat{p}(4)$. The relevant generalization for $N = (6, 8, ...)$ is straight forward.

For $S\widehat{p}(4)$, starting with (51)–(54) and (A9)–(A11) one obtains (as compared to (119))

$$\begin{aligned} |\Psi\rangle_{(n)} = (q^{-4} + q^{-2} + q^2 + q^4)^{-1/2}(q^{-2}\left|1\overline{1}\right\rangle \\ + q^{-1}\left|2\overline{2}\right\rangle - q\left|\overline{2}2\right\rangle - q^2\left|\overline{1}1\right\rangle), \end{aligned} \tag{124}$$

the corresponding entanglement entropy is

$$\begin{aligned} S(q) = S(q^{-1}) = \ln(q^{-4} + q^{-2} + q^2 + q^4) \\ -\frac{4q^4 + 2q^2 - 2q^{-2} - 4q^{-4}}{q^{-4} + q^{-2} + q^2 + q^4}\ln q. \end{aligned} \tag{125}$$

Thus again, as for $S\widehat{O}(4)$,

$$S(1) = S(\max) = \ln 4, \tag{126}$$

$$S(1 \mp \epsilon) = \ln 4 \pm 10\epsilon \ln(1 \mp \epsilon), \tag{127}$$

$$S(q) \to 0 \text{ as } q \to \infty \text{ or } q \to 0. \tag{128}$$

The slope, starting from $S(1)$ towards the asymptotic zero values is steeper as compared to the $S\widehat{O}(4)$ case, as shown in Fig. 1.

One can show (starting with P_0 defined in (13) and (14)) that for each N,

$$\begin{aligned} S(q, N) &= S(q^{-1}, N), \\ S_{\max}(q, N) &= S(1, N) = \ln N, \\ S(q, N) &\to 0 \text{ as } q \to \infty \text{ or } q \to 0. \end{aligned} \tag{129}$$

We do not present the explicit (straight forward) computations for general cases. But one aspect is worth pointing out.

We emphasized in related previous papers[5,15] that for our special solutions $(S\widehat{O}(N), S\widehat{p}(N))$ the projector P_0 and the braid matrix $\widehat{R}$ remain non-trivial for $q = 1$. This is a remarkable feature (absent in standard solutions for $SU(N)_q$, $SO(N)_q$, $Sp(N)_q$) as was emphasized in Section 3 of Ref. 5 and Section 2 of Ref. 15.

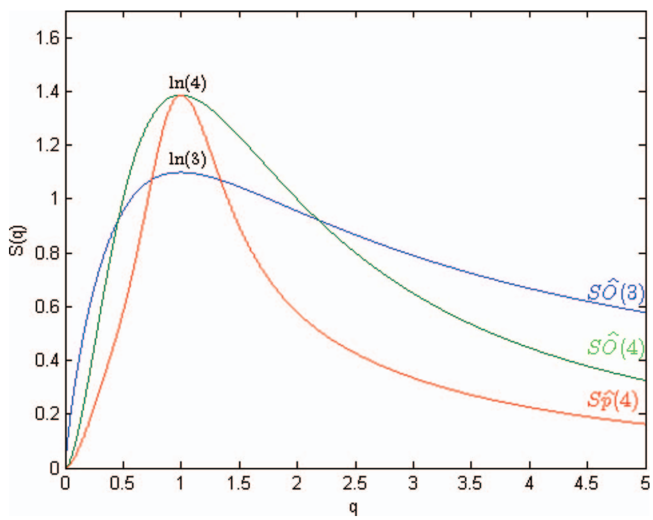

FIG. 1. (q, N)-dependent entanglement entropy.

Now we have found that not only $S\widehat{O}(N)$, $S\widehat{p}(N)$ remain non-trivial for $q = 1$, the associated entanglement entropy (for eigenstates of P_0) is maximal ($\ln N$) for $q = 1$. Thus, the striking non-triviality at $q = 1$ acquires further significance.

An adequate study of correlations in presence of multiple states $|\Psi\rangle$ (as already starting to appear in (88)) will not be undertaken in this paper. This aspect remains to be explored.

We have shown above that the operators P_0, acting on any $|ij\rangle$ either annihilates it (for $j \neq \bar{i}$) or generates (for $j = \bar{i}$) entangled states $|\Psi\rangle$ and we have quantified the corresponding entanglement for all (q, N) by computing the entanglement entropy.

VIII. CONCLUSIONS AND PERSPECTIVES

Starting with the projectors (P_+, P_-, P_0) for $SO_q(N)$ and $Sp_q(N)$ braid matrices and then keeping only P_0 it was shown how P_0 can generate a Temperley-Lieb algebra and how this property leads to a remarkable special class of braid matrices (denoted as $S\widehat{O}_q(N)$, $S\widehat{p}_q(N)$) and related spin chains.

Then we have explored certain aspects of such spin chains, using mostly $S\widehat{O}_q(3)$ examples of chains with free ends to display some particularly interesting properties.

Time evolution of such chains was studied by evaluating the actions of successive terms $(-\ iHt)^p$ in the series development of e^{-iHt}, H being the spin chain Hamiltonian. In particular, we studied in what form the data encoded in parameters of mixed states at one end of the chain can be decoded by observing mixed states reaching (as t increases) the other end of the chain. Most of the relevant computations has been collected together in Appendices A–C.

Finally, we have obtained the entanglement entropies $S(q,\ N)$ of the eigenstates of P_0. In particular we obtained (q, N)-dependence as

$$S(q, N) = S(q^{-1}, N), \tag{130}$$

$$S_{\max}(q, N) = S(1, N) = \ln N, \tag{131}$$

$$S(q, N) \to 0 \text{ as } q \to \infty \text{ or } q \to 0. \tag{132}$$

We pointed out before in Sec. IV that the two possible sign determinations of the essential parameter η correspond to two different regimes for the energy eigenvalues of the chain Hamiltonian. One may compare and contrast such a feature with the well-known corresponding ones of the 6-vertex models (see, for example, Ref. 18).

Certain aspects of our classes of spin chains remain to be studied, as pointed out in Sec. IV.

Another rich perspective is the exploration of various aspects of the braid matrices we started with (Secs. II and III) before extracting from them the chain Hamiltonian (Sec. IV).

In previous papers[13,19] we studied parametrized entanglements generated by braid operators rendered unitary by implementing imaginary rapidities ($i\theta$, $i\theta'$) in $\widehat{R}$ matrices of (5). Here again (from (8) and (9))

$$\begin{aligned}\widehat{R}(i\theta) &= P_+ + P_- + \frac{f_0\,(-i\theta)}{f_0\,(i\theta)} P_0 \\ &= I \otimes I + \omega\,(i\theta)\, P_0 \end{aligned} \tag{133}$$

can be directly verified to satisfy unitarity, i.e.,

$$(\widehat{R}_q(i\theta)^{\dagger} \widehat{R}_q(i\theta)) = I_N \otimes I_N. \tag{134}$$

Now one can try to formulate explicitly (q, N)-parametrized entanglement quantifiers of the superpositions of 3-qubit states generated by the action of the braid operator (see (5) and (6)) on such product states, as on the l.h.s. of (40) and (41) generalized to triple products. One can also examine possible teleportation protocols associated to our class of unitary matrices (see Ref. 2).

APPENDIX A: EXPLICIT P_0 ($N = 3, 4$)

Many basic results of Sec. V can be read off easily from the matrices P_0 presented below. The matrices (ij) are defined above (12) as are $(\overline{i\,j})$. The projectors P_0 are defined by (12)–(18). Their contents for the simplest cases are displayed below.

(i) $S\widehat{O}(3)$: ($N = 3$; $\bar{1} = 3$, $\bar{2} = 2$)

$$(q^{-1} + 1 + q)P_0 \equiv P_0'$$

$$\begin{aligned} &= q^{-1}(11) \otimes (\overline{1}\overline{1}) \\ &\quad + q^{-1/2}(12) \otimes (\bar{1}2) + (1\bar{1}) \otimes (\bar{1}1) \\ &\quad + q^{-1/2}(21) \otimes (2\bar{1}) + (22) \otimes (22) \\ &\quad + q^{1/2}(2\bar{1}) \otimes (21) + (\bar{1}1) \otimes (1\bar{1}) \\ &\quad + q^{1/2}(\bar{1}2) \otimes (12) + q(\overline{1}\overline{1}) \otimes (11), \end{aligned} \tag{A1}$$

$$= \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & q^{-1/2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1/2} & 0 & 1 & 0 & q^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & q^{1/2} & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}. \tag{A2}$$

Defining the base states

$$|ij\rangle \equiv |i\rangle \otimes |j\rangle\,,$$

the single eigenstate of P_0 with non-zero eigenvalue is

$$|\Psi\rangle \equiv q^{-1/2} \left|1\bar{1}\right\rangle + |22\rangle + q^{1/2} \left|\bar{1}1\right\rangle, \tag{A3}$$

$$P_0\,|\Psi\rangle = |\Psi\rangle\,. \tag{A4}$$

All 6 states $|ij\rangle$ with $j \neq i$ are annihilated by P_0. Also,

$$P_0(q^{1/2}\left|1\overline{1}\right\rangle - |22\rangle) = P_0(|22\rangle - q^{-1/2}\left|\overline{1}1\right\rangle) = 0. \tag{A5}$$

Corresponding patterns arise for all N. They will not be explored in any detail.

We present briefly the cases $N = 4$.

(ii) $S\widehat{O}(4)$: $(\overline{1} = 4,\ \overline{2} = 3)$

$$(q^{-2} + 2 + q^2)P_0 \equiv P_0'$$

$$\begin{aligned}
&= q^{-2}(11) \otimes (\overline{11}) + q^{-1}(12) \otimes (\overline{12}) \\
&+q^{-1}(1\overline{2}) \otimes (\overline{1}2) + (1\overline{1}) \otimes (\overline{1}1) \\
&+q^{-1}(21) \otimes (\overline{21}) + (22) \otimes (\overline{22}) \\
&+(2\overline{2}) \otimes (\overline{2}2) + q(2\overline{1}) \otimes (\overline{2}1) \\
&+q^{-1}(\overline{2}1) \otimes (2\overline{1}) + (\overline{2}2) \otimes (2\overline{2}) \\
&+(\overline{22}) \otimes (22) + q(\overline{21}) \otimes (21) \\
&+(\overline{1}1) \otimes (1\overline{1}) + q(\overline{1}2) \otimes (1\overline{2}) \\
&+q(\overline{12}) \otimes (12) + q^2(\overline{11}) \otimes (11).
\end{aligned} \tag{A6}$$

Defining

$$|\Psi\rangle \equiv q^{-1}\left|1\overline{1}\right\rangle + \left|2\overline{2}\right\rangle + \left|\overline{2}2\right\rangle + q\left|\overline{1}1\right\rangle, \tag{A7}$$

$$P_0\,|\Psi\rangle = |\Psi\rangle\,. \tag{A8}$$

(iii) $S\widehat{p}(4)$: $(\overline{1} = 4,\ \overline{2} = 3)$

$$(q^{-4} + q^{-2} + q^2 + q^4)P_0 \equiv P_0'$$

$$\begin{aligned}
&= q^{-4}(11) \otimes (\overline{11}) + q^{-3}(12) \otimes (\overline{12}) \\
&-q^{-1}(1\overline{2}) \otimes (\overline{1}2) - (1\overline{1}) \otimes (\overline{1}1) \\
&+q^{-3}(21) \otimes (\overline{21}) + q^{-2}(22) \otimes (\overline{22}) \\
&-(2\overline{2}) \otimes (\overline{2}2) - q(2\overline{1}) \otimes (\overline{2}1) \\
&-q^{-1}(\overline{2}1) \otimes (2\overline{1}) - (\overline{2}2) \otimes (2\overline{2}) \\
&+q^2(\overline{22}) \otimes (22) + q^3(\overline{21}) \otimes (21) \\
&-(\overline{1}1) \otimes (1\overline{1}) - q(\overline{1}2) \otimes (1\overline{2}) \\
&+q^3(\overline{12}) \otimes (12) + q^4(\overline{11}) \otimes (11).
\end{aligned} \tag{A9}$$

Defining

$$|\Psi\rangle \equiv q^{-2}\left|1\overline{1}\right\rangle + q^{-1}\left|2\overline{2}\right\rangle - q\left|\overline{2}2\right\rangle - q^2\left|\overline{1}1\right\rangle, \tag{A10}$$

$$P_0\,|\Psi\rangle = |\Psi\rangle\,. \tag{A11}$$

For $S\widehat{p}(N)$ the blocks with negative signs are anti-diagonally aligned.

APPENDIX B: ITERATIVE ACTION OF *H*

As explained in Sec. VI, in studying $S\widehat{O}(3)$ chains, it is useful to have ready results for

$$H^{m}_{(3)}(|\Psi\rangle\,|i\rangle\,,\,|i\rangle\,|\Psi\rangle)$$

and

$$H^{m}_{(4)}(|i\rangle\,|\Psi\rangle\,|j\rangle).$$

Here we collect the results indicating the derivations. We consider below $H'_{(3)}$, $H'_{(4)}$ as defined below (72) in "notation." Necessary multiplicative factors can be easily supplied. We start with results (73) and (74). Using them one obtains

$$\begin{aligned} H'_{(3)}(|\Psi\rangle\,|i\rangle) &\equiv (P'_0\otimes I + I\otimes P'_0)(|\Psi\rangle\,|i\rangle) \\ &= k\,|\Psi\rangle\,|i\rangle + |i\rangle\,|\Psi\rangle\,, \end{aligned} \tag{B1}$$

where $k=(q^{-1}\,+\,1\,+\,q)$, $i=(1,2,\overline{1})$ also

$$H'_{(3)}(|i\rangle\,|\Psi\rangle) = k\,|i\rangle\,|\Psi\rangle + |\Psi\rangle\,|i\rangle\,. \tag{B2}$$

Iterating

$$(H'_{(3)})^{p}(|\Psi\rangle\,|i\rangle) = A_p\,|\Psi\rangle\,|i\rangle + B_p\,|i\rangle\,|\Psi\rangle\,, \tag{B3}$$

$$(H'_{(3)})^{p}(|i\rangle\,|\Psi\rangle) = A_p\,|i\rangle\,|\Psi\rangle + B_p\,|\Psi\rangle\,|i\rangle\,, \tag{B4}$$

where

$$A_p = \frac{1}{2}((k+1)^p + (k-1)^p), \tag{B5}$$

$$B_p = \frac{1}{2}((k+1)^p - (k-1)^p). \tag{B6}$$

Now

$$H'_{(4)}(|i\rangle\,|\Psi\rangle\,|j\rangle) \equiv (P'_0\otimes I\otimes I + I\otimes P'_0\otimes I \tag{B7}$$

$$\begin{aligned} &+I\otimes I\otimes P'_0)(|i\rangle\,|\Psi\rangle\,|j\rangle) \\ &= |i\rangle\,|j\rangle\,|\Psi\rangle + |\Psi\rangle\,|i\rangle\,|j\rangle \\ &+k\,|i\rangle\,|\Psi\rangle\,|j\rangle \end{aligned} \tag{B8}$$

or

$$(H'_{(4)}-k)(|i\rangle\,|\Psi\rangle\,|j\rangle) = |i\rangle\,|j\rangle\,|\Psi\rangle + |\Psi\rangle\,|i\rangle\,|j\rangle\,. \tag{B9}$$

For $j\neq\overline{i}$ (when $i\neq\overline{j}$), $P'_0\,|ij\rangle=0$ and from (B1), (B2), and (B9)

$$(H'_{(4)}-k)^2(|i\rangle\,|\Psi\rangle\,|j\rangle) = 2\,|i\rangle\,|\Psi\rangle\,|j\rangle\,. \tag{B10}$$

Thus,

$$\begin{aligned} (H'_{(4)}-k)^{2n}(|i\rangle\,|\Psi\rangle\,|j\rangle) &= 2^{n-1}\,|i\rangle\,|\Psi\rangle\,|j\rangle\,, \qquad \text{(B11)} \\ (H'_{(4)}-k)^{2n+1}(|i\rangle\,|\Psi\rangle\,|j\rangle) &= 2^{n-1}(H'_{(4)}-k)\,|i\rangle\,|\Psi\rangle\,|j\rangle \\ &= 2^{n-1}(|i\rangle\,|j\rangle\,|\Psi\rangle + |\Psi\rangle\,|i\rangle\,|j\rangle). \end{aligned}$$

These results can be implemented directly by writing

$$e^{-iHt} = e^{-ikt}e^{-i(H-k)t},$$

and using the series development of the last factor. For $j = \bar{i}$ there are extra terms as follows:

$$(H'_{(4)} - k)(|i\rangle\,|\Psi\rangle\,|\bar{i}\rangle) = |i\bar{i}\rangle\,|\Psi\rangle + |\Psi\rangle\,|i\bar{i}\rangle\,. \tag{B12}$$

Hence

$$(H'_{(4)} - k)^2(|i\rangle\,|\Psi\rangle\,|\bar{i}\rangle) = 2\,|i\rangle\,|\Psi\rangle\,|\bar{i}\rangle + 2\,|\Psi\rangle\,|\Psi\rangle\,. \tag{B13}$$

But now

$$\begin{aligned} H'_{(4)}\,|\Psi\rangle\,|\Psi\rangle &= 2\,|\Psi\rangle\,|\Psi\rangle + (I \otimes P'_0 \otimes I)\,|\Psi\rangle\,|\Psi\rangle \\ &= 2\,|\Psi\rangle\,|\Psi\rangle + (q^{-1/2}\,|1\rangle\,|\Psi\rangle\,|\bar{1}\rangle \\ &\quad + |2\rangle\,|\Psi\rangle\,|2\rangle + q^{1/2}\,|\bar{1}\rangle\,|\Psi\rangle\,|1\rangle), \end{aligned} \tag{B14}$$

$$\begin{aligned} &H'_{(4)}(q^{-1/2}\,|1\rangle\,|\Psi\rangle\,|\bar{1}\rangle + |2\rangle\,|\Psi\rangle\,|2\rangle + q^{1/2}\,|\bar{1}\rangle\,|\Psi\rangle\,|1\rangle) \\ &= 2\,|\Psi\rangle\,|\Psi\rangle + k(q^{-1/2}\,|1\rangle\,|\Psi\rangle\,|\bar{1}\rangle \\ &\quad + |2\rangle\,|\Psi\rangle\,|2\rangle + q^{1/2}\,|\bar{1}\rangle\,|\Psi\rangle\,|1\rangle). \end{aligned} \tag{B15}$$

Combining (B14) and (B15) one can now iterate.

APPENDIX C: EXPLICIT RESULTS FOR A 6-CHAIN

We present below the iterated action of H' (up to fifth order, namely, $(H')^5$) on the free 6-chain states (for our $S\widehat{O}(3)$ case)

$$|x\rangle_1 \equiv |\bar{1}11111\rangle\,, \tag{C1}$$

$$|x\rangle_2 \equiv |1\bar{1}1111\rangle\,. \tag{C2}$$

They will be implemented in Sec. VI to study, explicitly for a simple case, the time evolution of our class of spin chains and possible data transmission with such evolutions.

Here the relevant H' is (with P'_0 defined in Sec. V)

$$H'_{(6)} = \sum_{l=1}^{5} I \otimes I \otimes \ldots \otimes (P'_0)_{l,l+1} \otimes \ldots \otimes I\,. \tag{C3}$$

The actions of P'_0 on $S\widehat{O}(3)$ states are defined in Sec. V and iterative actions are presented, for H' acting on $S\widehat{O}(3)$ states in Sec. VI. For the restricted case relevant here only one needs, for sub-chains of $H'_{(6)}$ above

$$H'_{(3)} \equiv P'_0 \otimes I + I \otimes P'_0, \tag{C4}$$

$$H_{(4)} \equiv P'_0 \otimes I \otimes I + I \otimes P'_0 \otimes I + I \otimes I \otimes P'_0, \tag{C5}$$

acting, respectively, on

$$H'_{(3)}(|\Psi\rangle\,|1\rangle\,, |1\rangle\,|\Psi\rangle) \tag{C6}$$

$$= ((k\,|\Psi\rangle\,|1\rangle + |1\rangle\,|\Psi\rangle), (|\Psi\rangle\,|1\rangle + k\,|1\rangle\,|\Psi\rangle)),$$

$$H'_{(4)}(|1\rangle\,|\Psi\rangle\,|1\rangle) = |\Psi\rangle\,|11\rangle + k\,|1\rangle\,|\Psi\rangle\,|1\rangle + |11\rangle\,|\Psi\rangle\,. \tag{C7}$$

Here we have used the basic definitions and results (42)–(46).

Using all these results systematically one obtains the following results in a straightforward fashion, arranging terms in the order shown below:

$$H'\,|x\rangle_1 = q^{1/2}\,|\Psi\rangle\,|1111\rangle\,, \tag{C8}$$

$$H'\,|x\rangle_2 = q^{-1}H'\,|x\rangle_1 + q^{1/2}\,|1\rangle\,|\Psi\rangle\,|111\rangle\,, \tag{C9}$$

where $|\Psi\rangle = q^{-1/2}\left|1\bar{1}\right\rangle + |22\rangle + q^{1/2}\left|\bar{1}1\right\rangle$.

$$(H')^2\,|x\rangle_1 = q^{1/2}(k\,|\Psi\rangle\,|1111\rangle + |1\rangle\,|\Psi\rangle\,|111\rangle), \tag{C10}$$

$$\begin{aligned}(H')^2\,|x\rangle_2 = {}& q^{-1}(H')^2\,|x\rangle_1 + q^{1/2}(|\Psi\rangle\,|1111\rangle \\ &+k\,|1\rangle\,|\Psi\rangle\,|111\rangle + |11\rangle\,|\Psi\rangle\,|11\rangle),\end{aligned} \tag{C11}$$

where $k = (q^{-1}\ +\ 1\ +\ q)$.

$$\begin{aligned}(H')^3\,|x\rangle_1 = {}& q^{1/2}((k^2+1)\,|\Psi\rangle\,|1111\rangle \\ &+2k\,|1\rangle\,|\Psi\rangle\,|111\rangle + |11\rangle\,|\Psi\rangle\,|11\rangle),\end{aligned} \tag{C12}$$

$$\begin{aligned}(H')^3\,|x\rangle_2 = {}& q^{-1}(H')^3\,|x\rangle_1 + q^{1/2}(2k\,|\Psi\rangle\,|1111\rangle \\ &+(k^2+2)\,|1\rangle\,|\Psi\rangle\,|111\rangle + 2k\,|11\rangle\,|\Psi\rangle\,|11\rangle \\ &+|111\rangle\,|\Psi\rangle\,|1\rangle).\end{aligned} \tag{C13}$$

$$\begin{aligned}(H')^4\,|x\rangle_1 = {}& q^{1/2}((k^3+3k)\,|\Psi\rangle\,|1111\rangle \\ &+(3k^2+2)\,|1\rangle\,|\Psi\rangle\,|111\rangle \\ &+3k\,|11\rangle\,|\Psi\rangle\,|11\rangle + |111\rangle\,|\Psi\rangle\,|1\rangle),\end{aligned} \tag{C14}$$

$$\begin{aligned}(H')^4\,|x\rangle_2 = {}& q^{-1}(H')^4\,|x\rangle_1 \\ &+q^{1/2}((3k^2+2)\,|\Psi\rangle\,|1111\rangle \\ &+(k^3+6k)\,|1\rangle\,|\Psi\rangle\,|111\rangle \\ &+(3k^2+3)\,|11\rangle\,|\Psi\rangle\,|11\rangle \\ &+4k\,|111\rangle\,|\Psi\rangle\,|1\rangle + |1111\rangle\,|\Psi\rangle).\end{aligned} \tag{C15}$$

$$(H')^5 |x\rangle_1 = q^{1/2}((k^4+6k^2+2)|\Psi\rangle|1111\rangle \\ +(4k^3+8k)|1\rangle|\Psi\rangle|111\rangle \\ +(6k^2+3)|11\rangle|\Psi\rangle|11\rangle \\ +4k|111\rangle|\Psi\rangle|1\rangle+|1111\rangle|\Psi\rangle), \tag{C16}$$

$$(H')^5 |x\rangle_2 = q^{-1}(H')^5|x\rangle_1 \\ +q^{1/2}((4k^3+8k)|\Psi\rangle|1111\rangle \\ +(k^4+12k^2+5)|1\rangle|\Psi\rangle|111\rangle \\ +(4k^3+13k)|11\rangle|\Psi\rangle|11\rangle \\ +(7k^2+4)|111\rangle|\Psi\rangle|1\rangle+5k|1111\rangle|\Psi\rangle). \tag{C17}$$

From the preceding results the coefficients (up to $O(t^5)$) of the states $|1111\rangle$ ($|\bar{1}1\rangle$, $|22\rangle$, $|1\bar{1}\rangle$) are obtained as given below:

$$e^{-i\lambda t H'}|X\rangle_1 = ... + |1111\rangle(x_1|\bar{1}1\rangle \tag{C18}$$

$$+y_1|1\bar{1}\rangle + z_1|22\rangle),$$

$$e^{-i\lambda t H'}|X\rangle_2 = ... + |1111\rangle(x_2|\bar{1}1\rangle \\ +y_2|1\bar{1}\rangle + z_2|22\rangle), \tag{C19}$$

and

$$x_1 = \frac{1}{4!}(\lambda t)^4 - \frac{i}{5!}(\lambda t)^5(4k+q)+O(t^6), \\ y_1 = -\frac{i}{5!}(\lambda t)^5 + O(t^6), \\ z_1 = -\frac{i}{5!}(\lambda t)^5 q^{-1/2} + O(t^6), \tag{C20}$$

$$x_2 = \frac{1}{3!}(\lambda t)^3 + \frac{1}{4!}(\lambda t)^4(4k+q+q^{-1}) \\ -\frac{i}{5!}(\lambda t)^5(7k^2+k(5q+4q^{-1})+5)+O(t^6), \\ y_2 = \frac{1}{4!}(\lambda t)^4 - \frac{i}{5!}(\lambda t)^5(5k+q^{-1})+O(t^6), \\ z_2 = -\frac{i}{5!}(\lambda t)^5 q^{1/2}(5k+q^{-2})+O(t^6). \tag{C21}$$

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