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REGULAR ARTICLE

Quantifying invariant features of within-group inequality in consumption across groups

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Abstract We study unit-level expenditure on consumption across multiple countries and multiple years, in order to extract invariant features of consumption distribution. We show that the bulk of it is lognormally distributed, followed by a power law tail at the limit. The distributions coincide with each other under normalization by mean expenditure and log scaling even though the data is sampled across multiple dimension including, e.g. time, social structure and locations. This phenomenon indicates that the dispersions in consumption expenditure across various social and economic groups are

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significantly similar subject to suitable scaling and normalization. Further, the results provide a measurement of the core distributional features. Other descriptive factors including those of sociological, demographic and political nature, add further layers of variation on the this core distribution. We present a stochastic multiplicative model to quantitatively characterize the invariance and the distributional features.

Keywords Inequality · Invariance · Consumption distribution · Power law · Lognormal distribution

JEL Classification D63 · D30 · I32 · E21

Any city, however small, is in fact divided into two, one the city of the poor, the other of the rich …
-Plato [380 BC]

1 Introduction

Plato's remark as stated at the beginning refers to the intuitive notion that even though the reason for inequality could be vastly different, but the dispersion in affluence is always present. Seminal work by Pareto (1897) shows that the right tail of the wealth distribution has a power law tail. Further explorations have shown that this feature is invariant across countries although the origin of the power law is not settled yet, either theoretically and empirically. The existence of a fat tail constitutes one such invariance even though the exponent and the share accruing to the top income classes are seen to be fluctuating substantially across countries and time (Atkinson and Piketty 2010; see also Chakrabarti et al. 2013). The bulk of the distributions are also described well by lognormal or gamma distributions (Chakrabarti et al. 2013), which again is susceptible to substantial variation in parameters even though the functional form remains the same. Thus there is hardly any precise and specific quantitative feature which is common across samples.

Quantifying inequality has been one of the most important factors in devising economic policies targeted towards mitigating the same. Theoretical tools developed for that purpose are equipped to find out the level of inequality based on a vector of income or wealth from a sample of units. Depending on the case, the unit could be an individual or a household (or something else). Keeping track of the level of inequality for the same sample over time or the same across different samples collected at the same point in time, allows us to make comparative judgments about the dynamics of inequality. In this paper, we ask the question: is there any fundamental feature of inequality that is invariant across samples (both across time and geographic boundary)?

We argue that in case of consumptions, the mean of the distribution is an important scaling factor. Once the distributions are normalized by their respective mean values, the inequality of the normalized sample show reasonable agreement in terms of numerical values. We study data with large sample size across three countries (India, Brazil and Italy) and a number of years (distinct waves when the data were collected).

Each country is a fiscal and monetary union of smaller states and/or other units e.g. religious or ethnic groups. The financial markets are also more integrated within each country than across countries. Both of these imply that the consumption decisions faced by households within a country are made in an environment much similar than households across countries. We show that within each country, the consumption distribution across different economic or social identities (states or religions or locations) show almost identical features once normalized by the respective mean values. The choice of set of countries (India, Brazil and Italy) under study stems from data availability.

To account for the distributional features, we provide a small-scale heterogeneous households model to quantify the dispersion and the existence of both lognormal bulk and power law tail. Previous models had either focused on the bulk which is lognormally distributed e.g. the literature that builds upon the approach proposed by Kalecki or the tail, which is power law distributed (see Chakrabarti et al. 2013). In the present paper, we propose a mixture model that is able to generate both simultaneously. In the model, we assume that households' consumption decisions are affected by habit. They interact through a capital market and receive idiosyncratic shocks in labor income that they cannot smooth out. Such incompleteness in the market along with heterogeneity in habits across households, generate a distribution of consumption. Using tools from distributional analysis, we show that the distribution has a dominant power law component in the limit and a lognormal bulk.

Methodologically, the idea of collapse of distributions with proper scaling of social data came from Fortunato and Castellano (2007) and Radicchi et al. (2008). Fortunato and Castellano (2007) proposed that such colapse is seen in voting behavior and Radicchi et al. (2008) showed that such behavior is also found in citation distribution. However, our work is probably the first one relating the collapse behavior with large-scale economic data. In terms of statistical features, we show that the tail of the distribution follows power law behavior which is seemingly absent in the social data analyzed in the above-mentioned papers. For the sake of completeness, we should also mention that in recent times the existence of power laws in income and wealth data have come under scrutiny (see e.g. Clauset et al. 2009). Brezinsky (2014) in fact proposes that power law fits only 35% of the wealth distribution of richest people (data obtained from Forbes' List; see paper for further details). However, we note that the existing evidence in favor of power law in the wealth distribution is comparatively much larger (see Chakrabarti et al. 2013; Sinha et al. 2010 for very comprehensive reviews). Thus we consider our result that consumption distribution has a power law tail, to be consistent with the current literature.

Chatterjee et al. (2016) was an initial attempt to study if there is any invariance in consumption. However, the scope of that study was very limited due to data availability (India 66th round, year 2009–2010). In the present context, we have analyzed a much bigger data set from multiple countries spanning over multiple years. This paper is related to two strands of literature. One, we invoke the idea of invariance in distributions of economic quantities like income or wealth (Pareto 1897). We differentiate our work from Kuznets (1955) who stressed the evolution of inequality across time due to evolution of market institutions. In a similar vain, Acemoglu and Robinson (2013) proposed a theory of historical evolution of inequality as a reflection of the evolution of political institutional features. A different version was proposed by Galor (2011) which emphasized development of institutions, specially educational sector being an important factor. Our approach is complementary in that we propose that there always exists a substantial level of inequality conditional on the state of the economy which is captured by the average affluence. On the technical side, there are multiple attempts to model the power law structure which is the most commonly known invariant feature of income/wealth distributions. Chakrabarti et al. (2013) contains a number of models in that direction. Benhabib et al. (2014) showed that it is possible to generate a power law in the tail by using an overlapping generation framework with incomplete markets. As we have discussed in the modeling section in details, we use the specifications in Gabaix (2011) and Kelly et al. (2013) for analytical purpose.

In the next section, we describe the data and summary statistics of consumption distribution. In the following section, we present the key results regarding variation in consumption across countries and time. Finally, to account for the robust pattern we see in data, we present a simple stochastic model of consumption distribution.

2 Data description

We use the data for Household Consumer Expenditure 68th Round (2011–2012) from the National Sample Survey Office NSSO (2011–2012) of India. It contains information about expenditure incurred by households on consumption goods and services during the reference period. These sample surveys are conducted using households as unit of the economy. This ignores heterogeneity in household size but the data contains information about monthly per capita consumer expenditure (MPCE) in Indian Rupee (INR). Data is available for all sampled households in the different states and Union territories (UT), across several parameters like castes, religions and rural–urban divide. Chatterjee et al. (2016) studied Household Consumer Expenditure 66th Round (2009–2010) collected by the National Sample Survey Office (NSSO) which collected data for multiple definitions of expenditure and used multiple definitions of inequality. The important conclusions we draw from that study is that the results are robust to such changes in definitions. So we focus on very specific and standard definitions only, in the present work. There are 101,717 households in this data set (see Tables 4, 5). To study the inequality structure, we use two kinds of data which provides two perspectives.

Data from Brazil is procured from IBGE—Instituto Brasileiro de Geografia e Estatística. The Consumer Expenditure Survey data from two rounds (IBGE (2002–2003) with 48,470 households and IBGE (2008–2009) with 55,970 households) are used here (Table 1). The data contains information about household size, geographical location (state) and consumer expenditure in Brazilian Real (BRL) for different expenditure sectors, among other things.

For Italy, we use microdata provided by d’Italia (2015) and has information about household consumer expenditure in Euro (EUR). We analyze 10 years of data

Table 1 Number of households, average per capita consumption expenditure $E(x)$ and Gini indices for Brazil, for 2 rounds 2002–2003 and 2008–2009

ID	State index		2002–2003			2008–2009		
			#Households	$E(x)$	Gini	#Households	$E(x)$	Gini
1	11	Rondônia	1112	10,870	0.535	907	15,090.8	0.498
2	12	Acre	960	8361.9	0.570	863	12,854.8	0.484
3	13	Amazonas	1075	7388.29	0.549	1344	11,011.6	0.504
4	14	Roraima	554	9221.05	0.529	644	12,965.1	0.558
5	15	Pará	1666	6645.31	0.509	1894	12,063.7	0.538
6	16	Amapá	568	7163.86	0.510	689	14,428.5	0.537
7	17	Tocantins	933	8027.05	0.569	1270	12,306.8	0.498
8	21	Maranhão	2231	4749.73	0.502	2562	8725.61	0.524
9	22	Piauí	2222	6263.12	0.557	2056	9844.81	0.498
10	23	Ceará	2017	6351.92	0.571	1861	8553.52	0.514
11	24	Rio Grande do Norte	1548	6819.38	0.558	1342	11,016.7	0.501
12	25	Paraíba	2367	5614.04	0.538	1628	10,990.6	0.543
13	26	Pernambuco	1674	7126.29	0.558	2367	11,360.2	0.532
14	27	Alagoas	2965	6601	0.583	2712	9138.34	0.541
15	28	Sergipe	1143	6705.08	0.518	1654	12,289.4	0.512
16	29	Bahia	2457	7678.18	0.584	3050	11,997	0.539
17	31	Minas Gerais	3004	11,283.4	0.528	5028	17,065.8	0.508
18	32	Espírito Santo	2337	12,148.5	0.535	3489	16,538.2	0.511
19	33	Rio de Janeiro	1285	17,973.1	0.591	1938	22,063.4	0.551
20	35	São Paulo	2017	16,074.3	0.516	3623	22,592.2	0.486
21	41	Paraná	2263	12,733.5	0.519	2477	17,829.8	0.470
22	42	Santa Caterina	1989	12,169.9	0.464	2029	23,447.8	0.498
23	43	Rio Grande do Sul	1850	14,370.6	0.534	2210	20,394.7	0.482
24	50	Mato Grosso do Sul	2541	9788.09	0.505	2247	17,071.5	0.498
25	51	Mato Grosso	2355	9241.06	0.513	2423	14,363.5	0.488
26	52	Goías	2356	9160.95	0.505	2686	16,997.5	0.523
27	53	Distrito Federal	981	26,497.2	0.590	977	26,081.3	0.564
		All Brazil	48,470	9626.94	0.568	55,970	12,777	0.533

Tabulated across states

(1980–1984, 1986, 1987,1989, 1991, 1993, 1995, 1998, 2000, 2002, 2004, 2006, 2008, 2010, 2012). See "Data summary" of appendix for description and summary statistics.

3 Results

In this section, we discuss the main results of data analysis. First feature is that the normalized data collapses on one single distribution. The second feature is that the distribution is lognormal followed by power law tail.

3.1 Invariance

The available consumption data shows a scaling property across time and countries. Consider a variable x which denotes consumption expenditure. Suppose it has a distribution $p_{it}(x)$ in cross section, in the i-th country, t-th period. We show that by choosing a suitable scaling parameter, the data collapses into one single aggregate distribution across different countries and years upon taking log transformation. That is, the scaled variable

$$X_{it} = \log\left(\frac{x_{it}}{\tau_{it}^{1/\kappa}}\right) \tag{1}$$

has a distribution

$$p_{it}(X) = p(X) \tag{2}$$

for all i and t. The parameters we chose, are mean consumption expenditure ($\tau_{it} = E(x_{it})$ where $E(\cdot)$ denotes expectation operator) and $\kappa = 1$. Collapse of data onto a single distribution indicates that there is a core inequality process which is generated and described by mechanisms similar across geographic boundary and time.

3.2 Normalized distributions

Consider a variable x following a lognormal distribution,

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log x - \mu_x)^2}{2\sigma_x^2}\right). \tag{3}$$

The mean of this distribution is

$$E(x) = e^{\mu_x + \frac{\sigma_x^2}{2}} \tag{4}$$

and the variance is given by

$$V(x) = \left(e^{\sigma_x^2} - 1\right) \cdot e^{2\mu_x + \sigma_x^2}. \tag{5}$$

Thus upon normalization by the mean, the new distribution has a mean of

$$E^{norm}(x) = e^0 = 1 \tag{6}$$

and variance,

$$V^{norm}(x) = \left(e^{\sigma_x^2} - 1\right). \tag{7}$$

By linearizing the exponential term in the variance we get

$$V(x) = \sigma_x^2 + 2\mu\sigma_x^2 + \sigma_x^4 \tag{8}$$

whereas for the normalized variable we have

$$V^{norm}(x) = \sigma_x^2. \tag{9}$$

The tail of the distribution is found to be power law or a Pareto distribution which can be represented as

$$p(x) \sim x^{-(1+\gamma)}. \tag{10}$$

By the nature of the distribution, power law is scale-free i.e. normalization of data which is power law distributed, leaves the distributional features unchanged.

The scaled distribution in all cases show a lognormal bulk and a power law asymptotically. This form has also been argued to be extracted from income and wealth data (Chakrabarti et al. 2013). The difference is that here, the exponent of the power law tail is much higher than that of income or wealth distributions indicating much faster rate of decay and lower inequality (see also Sinha 2006; Jayadev 2008; Subramanian and Jayaraj 2009 for detailed analysis on household wealth distributions in India). The finding that consumption is less dispersed than income is consistent with the available evidence (Christiano 1987). Finally, we note that Eq. 1 is a linear transformation of the original expenditure variable and hence the distributional features remain intact. Only the moments change due to this transformation. See Sect. 3.2 for a short description of the parametric features of the normalized distribution.

We present the distribution of the scaled expenditure variable in Fig. 1 in for Indian states and also for other dimensions including caste (panel b), religion (panel c) and urbanity (panel d). This figure with superimposition of the cross-sectional data shows that the distributions coincide under normalization which is consistent with the preliminary findings made by Chatterjee et al. (2016) for different wave of data collection. The bulk of the data fits with lognormal distribution and the tail fits with a power law. Figure 2 shows a similar data collapse in case of Brazil across all states in two given years and Fig. 3 shows for multiple years across all states. The data is fitted with a lognormal distribution. In Fig. 4, we present normalized Italian data across years. Similar to the Indian data set, the bulk fits with lognormal distribution and the tail is fitted with a power law.

The fitting have been done in Gnuplot software. We have used nonlinear least-squares (NLLS) Marquardt–Levenberg algorithm for all of the fitting exercises. We also report the asymptotic standard errors of the point estimates. For Indian data (NSSO round 68; Fig. 1), bulk of the distribution has been fitted with lognormal distribution with parameters $\mu = -0.30 \pm 0.03, \sigma = 0.52 \pm 0.03$. The tail has been fitted with a Power law with coefficient -3.56 ± 0.02. For Brazilian data, we obtained $\mu = -0.65 \pm 0.01, \sigma = 0.95 \pm 0.01$ for the lognormal fitting (Fig. 3). For data obtained from Italy (Fig. 4), the bulk has been fitted with a lognormal distribution $\mu = -0.13 \pm 0.02, \sigma = 0.50 \pm 0.01$. The tail is fitted with a Power law with coefficient -4.3 ± 0.1. For robustness checks, we estimated the power-law coefficients with the methods developed by Clauset et al. (2009). This method gives joint estimation of the power law coefficient along with the cut-off. For India, the coefficient is 3.6 with an estimated cut-off of 4481. For Italy, the coefficent is 5.0 with an estimated cut-off of 75,000.

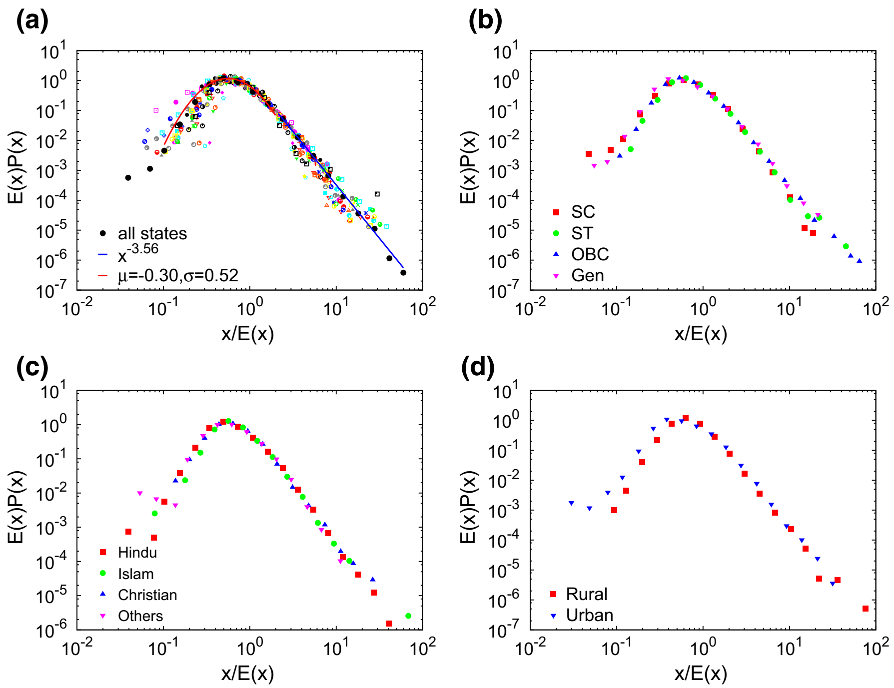

Fig. 1 Collapse of consumption expenditure data for Indian states. Data collected for round 68 (2011–2012). **a** Normalized data for all states. Fitted with a lognormal distribution and a power law at the right tail. **b** Normalized data across caste categorization. **c** Normalized data across religious categorization. **d** Normalized data across urban and rural population

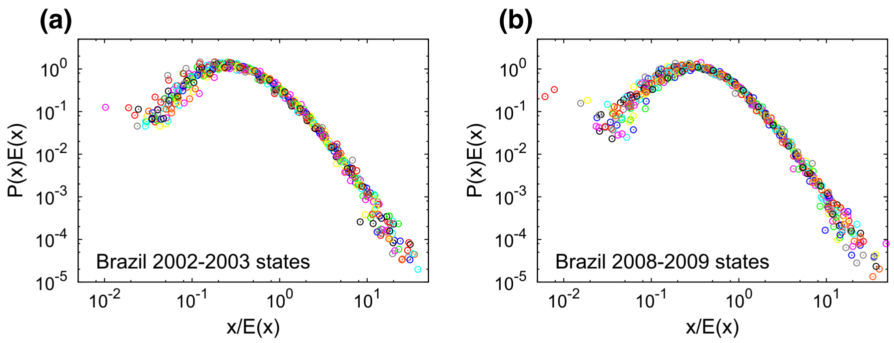

Fig. 2 Consumption expenditure data across Brazilian states normalized with respect to the respective mean expenditure across states; **a** year 2002–2003, **b** year 2008–2009. See Table 1 for details

We have executed formal tests of how close the distributions are across states or across years, after proper normalization. The results have been shown in Fig. 5. It should be noted that the variable under consideration is $\log(x/E(x))$ (Eq. 1). We show that for a substantial number of cases, the normalized and log-transformed variables across states within a country (India and Brazil) or across time for the same country

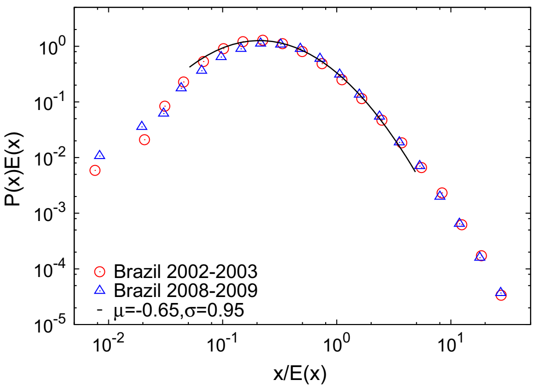

Fig. 3 Consumption expenditure data across years in all states of Brazil combined. **a** Year 2002–2003, **b** year 2008–2009

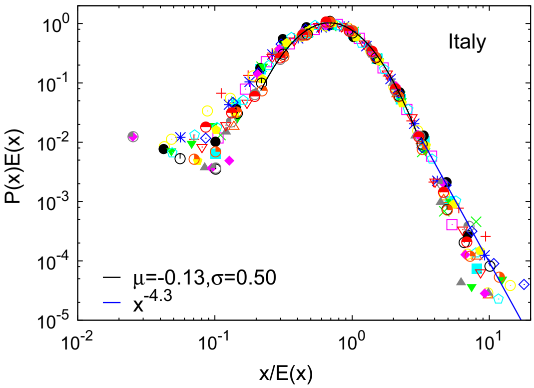

Fig. 4 Consumption expenditure data across years in all states of Italy combined. Data ranges from 1980–2012 (see Table 3 in "Appendix")

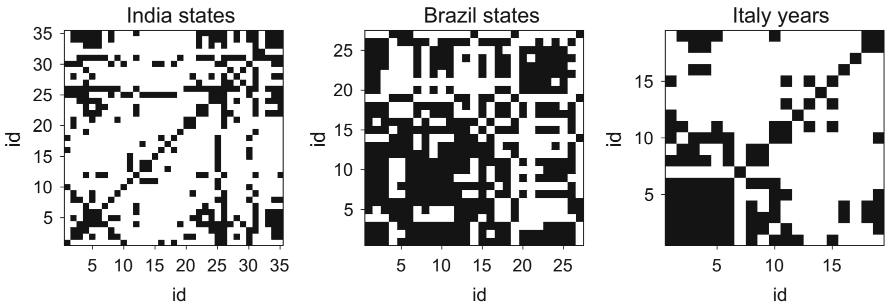

Fig. 5 Two sample Kolmogorov–Smirnov test for pairs of data. *Left panel*: pairwise test for all Indian states (2011–2012, Id: state). *Middle panel*: pairwise test for all Brazilian states (2008–2009, Id: state). *Right panel*: pairwise test for all Italian data across years (Id: year). Null hypothesis: two distributions are exactly identical. Colorcode: *white square* represents that the test rejects the null hypothesis at 1% level of significance. *Black squares* represent the complementary scenario

(Italy) are actually distributionally identical. Thus not only the broad algebraic forms of the distributions (lognormal bulk and power law tail) coincide, but also the parameters describing the distributions are very similar. In particular the Brazilian data establishes our claim. One interesting point is that such tests could be sensitive to existence of fat tails. Given that both India and Italy show prominent fat tails, it is not surprising that there are not many cases where the distributions are identical according to the test. Essentially this could be attributed to the existence of outliers (data on extreme right tail of the distribution) whereas the bulk of the data do fall on a single distribution.

4 An heterogeneous agent model

Here we propose a brief model of evolution of the consumption distribution. The basic goal would be to account for two observations, viz., the bulk of the distribution is seen to be following a lognormal distribution and there is a Pareto tail. Several assumptions are necessary to simplify the exposition. Time is discrete and goes till infinity i.e. $T = 1, 2, \ldots$. There are N dynasties who are producing and consuming. With a little abuse of notation, we will also use the same N to denote the set of agents as well where no confusions arise. Each dynasty can be thought of as an unit of observation in the present context. We do not attempt to provide any microfoundation of their consumption decision and construct our model based on the approach recently introduced in Gabaix (2011) and Kelly et al. (2013). They consider firm growth process resulting from interconnection among a large number of firms. The dynamical properties are developed from the proposed set of interconnections. In this case, we follow a similar route and assume that the growth rate of consumption expenditure at the unit level (the unit could be individual or family or household depending on the case) admits interconnections between agents who differ in their attitude towards consumption. At the same time, we keep our model general enough to incorporate aggregate effects like long-term growth which can potentially affect inequality (positively or negatively). Hence, the growth rate of consumption expenditure is assumed to be a function of the level of present expenditure, household specific factors and the state of the macroeconomy.

In particular, we propose the following behavioral form of growth rate of expenditure of the i-th unit at any generic time-point t

$$\begin{aligned}\hat{x}_i(t) &= \frac{\Delta x_i(t)}{x_i(t-1)},\\ &= \lambda_i(t)\,(x_i(t-1))^{\alpha_i(t)} + \frac{\eta_i(t) + r(t)b_i(t)\left(\sum_j^N w_j(t)\right) + \chi_i(t)}{x_i(t-1)} - 1 \qquad (11)\end{aligned}$$

where α_i is agent specific shock, $\eta_i(t)$ is the contribution of all unit-specific behavioral factors (e.g. religion, sex, geographic location etc.) that can potentially affect the level of consumption expenditure, $\chi_j(t)$ is a noise term with mean μ and variance σ^2. The term $\alpha_i(t)$ plays an important role to differentiate the households according to the propensity of multiplicative return (in log). We will elaborate on this point in Case I, II and II described below. The term in the middle requires elaboration. We assume that consumption expenditure is affected through wealth accumulation process. We are agnostic about the preferences of agents who participate in the same process and introduce a parameter $b_i(t)$ that captures the effects on the i-th dynasty. The rate of return is given by $r(t)$. The aggregate wealth (capital) is given by $\int w_j(t)dj$. below, we describe in details about the aggregate effect on consumption growth.

There are some classic studies on the growth rates of consumption. In particular, Hall (1978) provided a framework to study the growth rates of consumption expenditure in the following from,

$$\hat{x}_i(t) = \frac{\chi_i(t)}{x_i(t-1)}. \tag{12}$$

In "Random walk in consumption" of appendix we provide the basic framework that gives rise to such a growth rate. However, empirical studies have rejected such models. Jaeger (1992) presents evidence that the theory is rejected when tested with U.S. data. Haug (1991) shows that the discrepancy might come from time aggregation bias (see also Molana 1991). In a separate field of study, the firm growth rates had been described by similar functional forms. In particular, Gabaix (2011) starts describing a *granular* economy where each firm has a growth rate of

$$\hat{x}_i(t) = \chi_i(t) \tag{13}$$

which is known to generate a lognormal distribution (see below). Such specifications are known as Gibrat's law (Gibrat 1931). However, empirical estimations show that such a growth equation is incorrect (Hall 1987; Evans 1987; Calvo 2006). Kelly et al. (2013) expands this simple framework to incorporate relationships between growth rates as follows,

$$\hat{x}_i(t) = f(\{\hat{x}_j(t)\}_j, W_{i,j\in N}(t), \chi(t)) \tag{14}$$

where $f(\cdot)$ captures a linear evolution of growth rates across size x and W captures the interaction matrix. We combine the above mechanisms to propose Eq. 11.

Imagine that the production function is given by the following simple equation

$$Y(t) = s(t)K(t) \tag{15}$$

which says output is a linear function of capital (K) and a productivity shock s. One can incorporate labor. But for simplicity of exposition, we ignore it and assume that households (or individuals) supply labor inelastically. The rental payment in the competitive market exhausts the total output. Noting that wealth acts as capital, we see that the income is given by $Y(t) = r(t)\int w_j(t)dj$ in absence of wage income. Note that this also introduces a coupling among the agents through the capital market. This is the only source of direct interaction among agents (see Vikram and Sinha 2011 for a similar form of mean-field coupling to model asset fluctuation behavior in a multi-agent setting). Effectively such a factor induces an aggregate shock (rate or return r is a function of the aggregate shock s) to the dynamic process. This type of approach to link the dynamic processes of multiple agents can also be seen in Solomon (1998) although the links in the generalized Lotka–Volterra type models were considered for scaling purposes (and not for representing any aggregate shock).

We can decompose the income as the sum of a trend component ($Y^T(t)$) and a transitory component ($Y^C(t)$):

$$Y(t) = Y^T(t) + Y^C(t). \tag{16}$$

The transitory component captures the purely fluctuating part. Note that by construction

$$E(Y^C(t)) = 0. \tag{17}$$

Since we have shown that there is an invariance in the distribution of expenditure once the data is normalized with respect to the household specific microeconomic factors, we ignore their contribution for modeling the core inequality process. Also, we are ignoring the trend assuming that it does not affect inequality in the short-run. By taking all of the above into consideration, we arrive at the following equation,

$$\hat{x}_i(t) = \lambda_i(t)\,(x_i(t-1))^{\alpha_i(t)} + \frac{\big(b_i(t)Y^C(t) + \chi_i(t)\big)}{x_i(t-1)} - 1. \tag{18}$$

Therefore, we can use the definition of the growth rate and rewrite the above equation as

$$x_i(t) = \lambda_i(t)\,(x_i(t-1))^{1+\alpha_i(t)} + b_i(t)Y^C(t) + \chi_i(t). \tag{19}$$

The business cycle component induces a distortion on the mechanism. Note that it is a common factor to all agents. Thus if for a sustained period (a few quarters), the economy is either hit by very high or very low shocks expenditure growth rate is affected exacerbating inequality. On the other hand when the economy returns to the baseline (zero shock), the growth rates are diminished reducing inequality. Business cycle can be taken to be an exogenous factor as usually it affects inequality and the converse is unlikely. Thus due to the business cycles, inequality might wax and wane (Heathcote and Perri 2015 for example relates wealth inequality to GDP volatility).

Equation 19 forms the basis of the subsequent analysis. Below we show known solutions of the above dynamic equation in three limits.

Case I. Let the transitory component and the agents' idiosyncratic shocks to expenditure be identically equal to zero i.e. $Y^C(t) = 0$, $\alpha_i(t) = 0$ and $\chi_i(t) = 0$ for all t in Eq. 19. Then the equation boils down to the following form

$$\log(x_i(t)) = \log(x_i(t-1)) + \log\lambda_i(t). \tag{20}$$

This random walk in logarithm is known to generate a lognormal distribution (assuming the error term has mean μ_λ and standard deviation σ_λ),

$$f(x,t) = \frac{1}{x(t)\sqrt{2\pi\sigma_\lambda^2(t+1)}}\exp\left(-\frac{(\log x(t) - (t+1)\mu_\lambda)^2}{2\sigma_\lambda^2(t+1)}\right). \tag{21}$$

Due to the explicit time dependence, there does not exist a steady state distribution for this process. Standard deviation increases without bound over time at the rate $\sqrt{t+1}$ (see for example Chakrabarti et al. 2013).

Case II. Suppose the transitory component and the noise term are zero and the idiosyncratic term has a distribution over $[\alpha_{min}, \alpha_{max}]$ where $0 < 1 + \alpha < 1$ for all $\alpha \in [\alpha_{min}, \alpha_{max}]$ and all moments exist. Then we have

$$\log(x_i(t)) = (1 + \alpha_i(t)) \log(x_i(t - 1)) + \log \lambda_i(t). \tag{22}$$

By solving it recursively and using the lag operator L, we can rewrite it as

$$x_i(t) = \exp([1 + (1 + \alpha_i)L + (1 + \alpha_i)^2 L^2 + \cdots].\lambda_i(t)). \tag{23}$$

For simplicity of exposition we assumed $\alpha_i(t) = \alpha_i$ which can be relaxed without changing the basic result. Thus the bracketed term becomes the sum of an infinite series of noise terms with standard deviation going to zero in limit. Hence, this process reaches a steady state described by a lognormal distribution. This process was formulated and proposed as a model of income evolution by Kalecki in 1945 (see Chakrabarti et al. 2013 for a detailed exposition).

Case III. Consider Eq. 19 with the idiosyncratic term distributed over $[\alpha_{min}, \alpha_{max}]$ where $E(\alpha) = 0$ and $\sigma_\alpha \rightarrow 0$. We make two additional assumptions, (a) $E(\log \lambda_i(t)) < 0$ and (b) $\eta_t = b_i(t)Y^C(t) + \chi_i(t)$, is distributed over $\mathbb{R}_+$. Sornette and Cont (1997) shows that under such assumptions, the steady state distribution is power law. Such a dynamics is called Kesten process

$$x(t) = \lambda(t)x(t - 1) + \xi(t), \tag{24}$$

which is known to generate power laws in the limit (Kesten 1973). Gabaix (1999) uses such a mechanism to generate a power law in the city size distribution. See also Sornette (2006) for a textbook treatment.

4.1 Heterogeneity of agents

We introduce heterogeneity among the agents along one dimension viz., the upper range of the multiplicative factor (λ_{max}). Let us assume without loss of generalization that $0 < \lambda_{i,max} \leq \lambda_{j,max}$ for all $i \leq j$ and there exists some agent $1 < k < N$ for whom $\lambda_{k,max} = 0$. For all agents, $-1 < \alpha_{min} \leq \alpha_{max} \leq 0$. Thus effectively there are two types of agents. Fraction f of total number of agents have $0 < \lambda_{i,max} < 1$ for all $i \in N_f$ where N_f is the set of all such agents. The evolution of the expenditure is given by Eq. 19, which we can rewrite as

$$\log(x_i(t)) = (1 + \alpha_i(t)) \log(x_i(t - 1)) + \log \lambda_i(t) \tag{25}$$

ignoring the noise factor and the business cycle variation. Thus we are back to Case II above.

The second type is described by $\lambda_{i,max} > 1$ with the condition that $E(log(\lambda_i) < 0)$ as mentioned in Case III above. To gain intuition about why this process converges to a power law, assume that $E(\alpha) \rightarrow 0$ and $\sigma_\alpha \rightarrow 0$. We assume that $b_i(t)$ is highly procyclical i.e. $E(b_i, Y_C) > 0$. The first assumption allows us to maintain the exact parametric requirement of the Kesten process. Strong procyclicality of consumption share effectively induces a lower bar on the expenditure even when the variable receives consecutive bad shocks through α_i. Thus this becomes a reflective barrier and we can apply the methodology devised in the literature to find out the steady state distribution.

Gabaix (2009) provides a very simple proof that the mechanism generates a power law. Assuming the existence of the lower (reflective) boundary through the business cycle effects, we know that the variable can never be less than that. Hence, we consider the other extreme and study the right tail when the variable is far from the boundary making the additive terms relatively unimportant. Let us assume that the multiplicative factor λ_i is distributed according to $f(\lambda)$. Then we can write the evolution equation of the expenditure variable X as

$$Prob.\,(X_i(t) < x) = Prob.\left(X_i(t) < \frac{x}{\lambda_i(t)}\right) \tag{26}$$

Letting the left hand side be denoted as $M_t(e)$, we have a recursive equation

$$M_{t+1}(x) = \int_{\mathbb{R}_+} M_t\left(\frac{x}{\lambda}\right) f(\lambda) d\lambda. \tag{27}$$

The trick is to apply the criteria that when the system converges, the above equation would be time independent and one can guess and verify the functional forms. In particular, Gabaix (2009) shows that $M^{\infty}(X) \propto 1/X^{\gamma}$ solves the equation and the condition reduces to

$$E\left(\lambda^{\gamma}\right) = 1. \tag{28}$$

The same can also be shown using techniques developed by Sornette and Cont (1997). Equation 28 describes the relationship between the distribution of the multiplicative factor and the exponent of the distribution.

In Fig. 6, we present numerical simulations results for the evolution of the Pareto exponent in the more general context (Eq. 19). Equation 28 gives the solution in only one limit ($\alpha \to 0$). The left panel shows the determination of the exponent following Eq. 28 in the limit. We use Monte Carlo simulation to find the Pareto exponent γ in the general case. The right panel shows the estimated exponents for (α, $E(\lambda)$) pairs on the parameter plane. For the purpose of simulation, α is taken to be a constant. For each combination of α and $E(\lambda)$, we simulate Eq. 19 and estimate the exponent. The estimated exponents are averaged over $\mathcal{O}(10)$ realizations in order to arrive at stable values. The surface indicates the exponents for different pairs of α and $E(\lambda)$.

4.2 Shape of the ensemble distribution

As we have described in Sect. 4.1 above, there are essentially two types of agents. The first type generates a lognormal bulk whereas the second type generates a power law distribution for the tail. Here we want to show that for the aggregate distribution over all agents in the economy, the tail is indeed described by a power law.

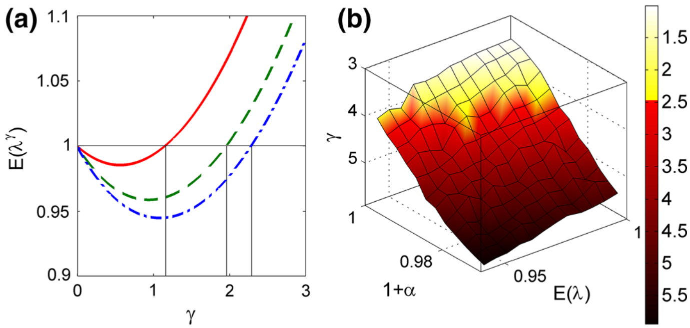

Fig. 6 Simulation results. **a** We plot the curve $E(\lambda^{\gamma})$ for 3 Monte Carlo realizations of the multiplicative shock λ (with different averages) over a range of γ. At the points of intersection with the horizontal line, one can find the theoretical prediction of the power law exponent γ from Eq. 28. **b** Estimated Pareto exponent γ for simulations over $(\alpha, E(\lambda))$ parameter space. *Color bar* shows the magnitude of γ with a cut-off at 2.5 which is indicative of the empirically estimated coefficient of the consumption distribution for India

For this purpose, we use a result (see Gabaix 2009 for a review) on the sum of two variables both distributed according power laws with potentially different exponents. Let the variables be v_1 and v_2. We assume that

$$x_i \sim C \cdot x_i^{-(1+\gamma_i)} \tag{29}$$

with $\gamma_1 \neq \gamma_2$. Then the sum these two variables ($x = x_1 + x_2$) will be distributed as

$$x \sim \bar{C} \cdot x^{-(1+min(\gamma_1,\gamma_2))}. \tag{30}$$

The intuition is simple: the fatter tail dominates the distribution. Note that the tail of a lognormal distribution can be approximated well by a power law with high exponent. Thus the tail of the aggregate distribution can be modeled as the sum of two power laws with different exponents. Since the exponent of the distribution that approximates a lognormal distribution is typically quite large, the other distribution dominates (following Eq. 30).

5 Discussion and summary

In this paper, we describe two robust features of consumption inequality across time and countries. One, if consumption data is normalized with a proper scaling factor, all data collapses on one single aggregate distribution. Two, the distribution has a lognormal bulk and a power law at the limit with high exponent (compared to income and wealth). Finally we provide a stochastic model to account for the basic distributional features.

In the present work, we differentiate between long run versus short run inequality. We focus exclusively on the latter in order to study cross sectional properties of inequality.

Throughout our analysis, we have considered nominal data. There are two reasons for it. One, the available data is in nominal terms. Two, in our cross-sectional analysis, we normalize the data first with respect to a scaling factor. As long as within region dispersion in price-levels are not significantly high, such a normalization takes care of the pricing factors. All the subsequent analysis including cross-time and cross-country comparisons are based on normalization with respect to respective scaling factor. Hence, such comparisons are free of biases due to between-region or between-time periods variations in general price levels.

The way we have described the consumption growth process, a number of additional implications can be presented. First, the power law arises due to the effects of business cycle implying that inequality in cross-section can be affected by business cycles. Heathcote and Perri (2015) documents that in U.S. mean wealth is negatively correlated with macroeconomic volatility. One can argue that the changes in mean wealth is also accompanied by redistribution of purchasing power affecting inequality, thus corroborating the prior implication. Stiglitz (2012) makes a point that there is a relationship between fluctuations of macroeconomic fundamentals and inequality. Secondly, in terms of the generative mechanism, our approach has a parallel with the method used by Benhabib et al. (2014) which also generates a power law in income in an overlapping generations framework. However, they provided a microfounded framework for consumption-savings decision even though the essential mechanism is similar. Earlier empirical works show that volatility of the business cycle is negatively related to the total income of the country (Canning et al. 1998). In terms of the model presented above, such a linkage would contribute to lower consumption volatility. In a fully specified utility-maximization framework this would imply higher welfare. Finally, all other non-economic factors are seen to affect the mean of the expenditure distribution. This has a corollary that the spread of the core inequality process is independent of the social, political and geographic factors. Angle (1992) and Angle (1993) also made a similar observation by considering U.S. data for specific social groups and conditioned on specific (e.g. racial or educational) factors. This is complementary to our approach where we focus exclusively on the idea of core features of the distribution.

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Appendix

In this section, we present the additional figures and tables. A simple derivation of random walk model in consumption is also presented.

Data summary

In Fig. 7, we present the available cross-sectional data for India for all states for three waves of data collection (2004–2005, 2009–2010 and 2011–2012). In the main text, we have analyzed data for 2011–2012. Chatterjee et al. (2016) presents some

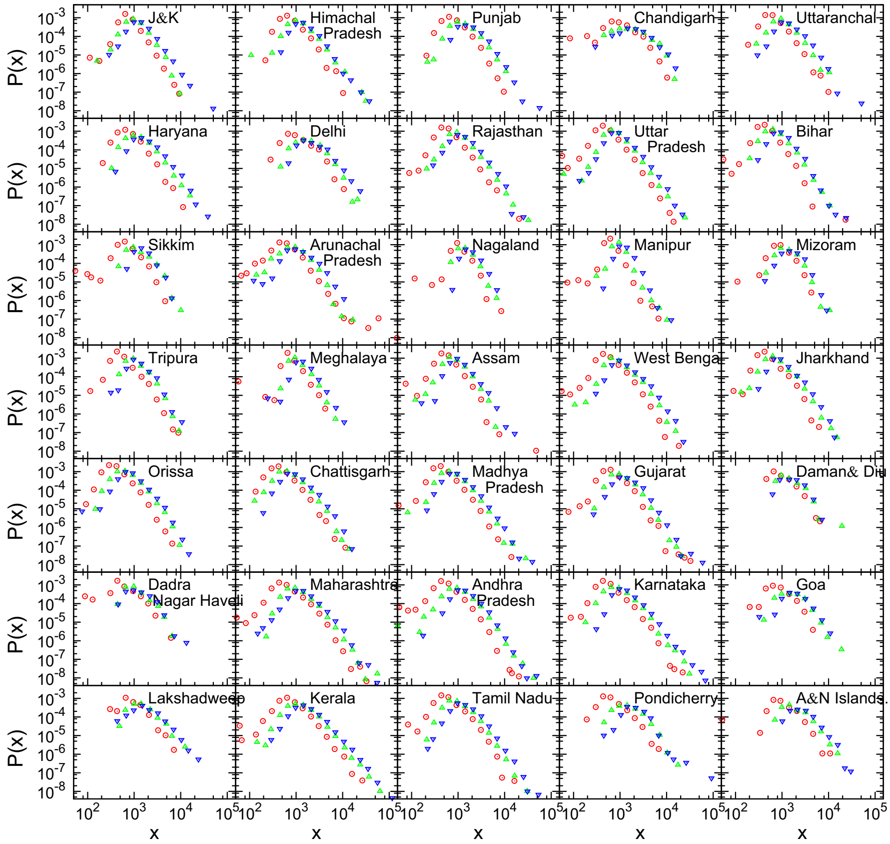

Fig. 7 Cross-sectional unnormalized data of Indian consumption expenditure across states has been shown for 3 waves of data collection (*red circles* for 2004–2005, *green upward triangles* for 2009–2010, *blue downward triangles* for 2011–2012). Average growth in consumption over time is evident

complementary results on the data set from 2009–2010. Figure 7 shows the general shift of the consumption density indicating both inflation and rising consumption power. Figure 8 shows the available Brazilian data for all states for two waves of data collection (2002–2003 and 2008–2009).

Details of the data and summary statistics have been tabulated in Table 1 (across states) and Table 2 (urban–rural). Table 3 contains summary statistics for the Italian data Finally, Table 4 (across states) and Table 5 (social and other dimensions) contains summary statistics for the Indian data.

Random walk in consumption

In a standard utility-maximizing framework with representative agent, the Euler equation would be

$$u'(x_t) = \beta R(t) E_t(u'(x_{t+1})) \tag{31}$$

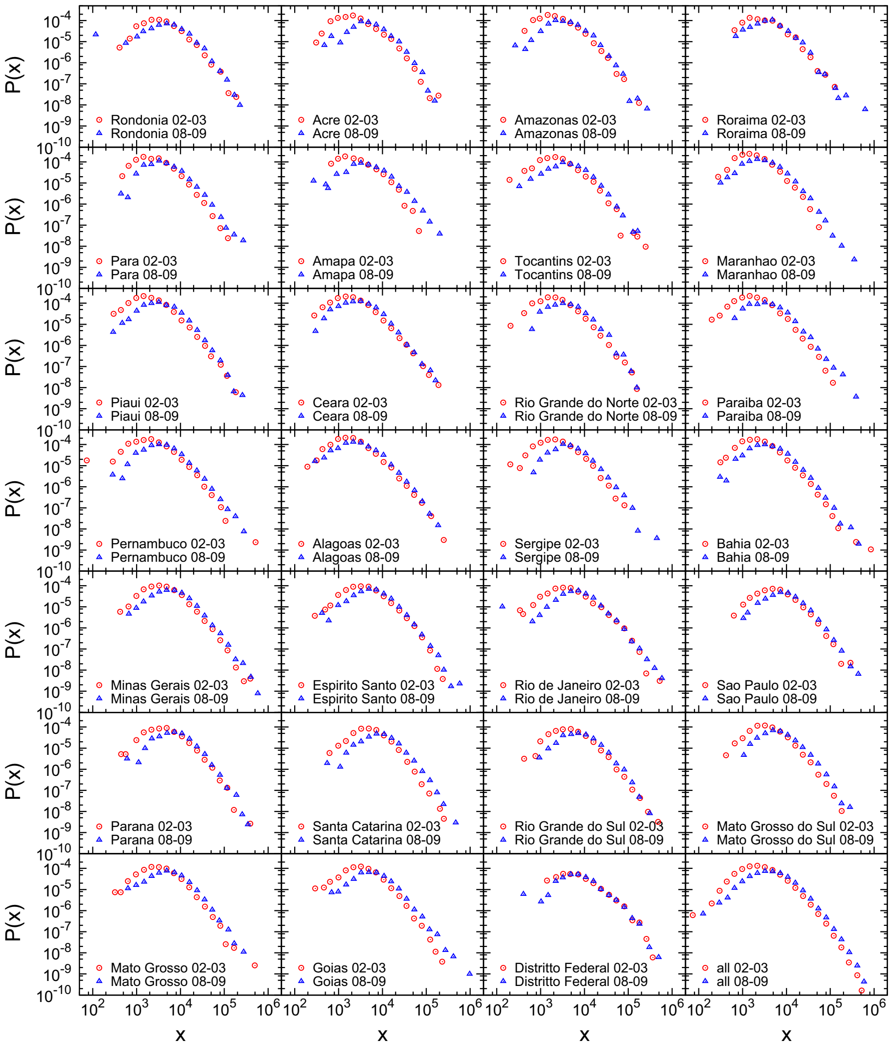

Fig. 8 Cross-sectional unnormalized data of consumption expenditure has been shown for 2 waves of data collection for different states in Brazil (*red circles* for 2002–2003, *blue upward triangles* for 2008–2009)

Table 2 Number of households, average per capita consumption expenditure $E(x)$ and Gini indices for Brazil, for 2 rounds 2002–2003 and 2008–2009

Location	2002–2003			2008–2009		
	#Households	$E(x)$	Gini	#Households	$E(x)$	Gini
Rural	48,357	18,352.6	0.514	43,193	16,658.7	0.528
Urban	114	20,352.3	0.478	12,777	9847.44	0.507

Tabulated according to location

Table 3 Number of households, average per capita consumption expenditure $E(x)$ and Gini indices for Italy, for several years

ID	Year	#Households	$E(x)$	Gini
1	1980	2980	8529.86	0.307
2	1981	4091	10,373.1	0.298
3	1982	3967	12,304.9	0.296
4	1983	4107	13,952.4	0.296
5	1984	4172	15,474.4	0.302
6	1986	8022	17,315.6	0.297
7	1987	8024	23,652.2	0.333
8	1989	8274	24,392.9	0.289
9	1991	8186	26,334.6	0.285
10	1993	8088	29,087.4	0.297
11	1995	8135	33,631.8	0.305
12	1998	7146	36,157.1	0.316
13	2000	8001	38,089.7	0.308
14	2002	8010	20,466.6	0.317
15	2004	8011	22,419.9	0.305
16	2006	7768	23,674.8	0.290
17	2008	7976	23,817.4	0.279
18	2010	7950	25,261.1	0.294
19	2012	8149	25,408.3	0.291

where $u(\cdot)$ is the utility function defined over consumption good x at time t. The discount factor is denoted by β and the rate of return by R. $E_s(\cdot)$ denotes expectation with the information set s. The simplest framework to derive random walk (Hall 1978) is to assume

$$u(x) = -\frac{(\bar{x} - x)^2}{2} \tag{32}$$

where $\bar{x}$ is the bliss point. Also assume $R\beta = 1$ to solve the above equation to get

$$E_t(x_{t+1}) = x_t. \tag{33}$$

Thus the consumption growth equation is

$$x_{t+1} = x_t + \chi_{t+1}, \tag{34}$$

where χ_{t+1} is the innovation term. Thus the growth rate is

$$\hat{x}(t) = \frac{\chi(t)}{x(t-1)}. \tag{35}$$

Table 4 Number of households, average per capita consumption expenditure $E(x)$, Gini index for India

ID	Geographic location	#Households	$E(x)$	Gini
1	Jammu and Kashmir	3382	1846.749	0.310
2	Himachal Pradesh	2040	2105.473	0.336
3	Punjab	3118	2571.475	0.334
4	Chandigarh	312	3577.070	0.378
5	Uttaranchal	1784	2073.443	0.350
6	Haryana	2589	2575.453	0.365
7	Delhi	999	3653.659	0.382
8	Rajasthan	4127	1824.600	0.332
9	Uttar Pradesh	9018	1414.226	0.357
10	Bihar	4581	1243.082	0.286
11	Sikkim	768	1850.008	0.243
12	Arunachal Pradesh	1674	1863.248	0.371
13	Nagaland	1024	2185.466	0.241
14	Manipur	2560	1438.841	0.220
15	Mizoram	1536	2129.177	0.259
16	Tripura	1856	1609.395	0.290
17	Meghalaya	1260	1759.212	0.263
18	Assam	3440	1417.833	0.309
19	West Bengal	6317	1886.182	0.387
20	Jharkhand	2737	1349.507	0.341
21	Orissa	4029	1246.751	0.347
22	Chattisgarh	2173	1464.659	0.367
23	Madhya Pradesh	4718	1449.213	0.366
24	Gujarat	3430	2143.533	0.345
25	Daman and Diu	128	2196.510	0.273
26	Dadra and Nagar Haveli	192	1901.413	0.335
27	Maharashtra	8041	2323.568	0.391
28	Andhra Pradesh	6898	2094.464	0.345
29	Karnataka	4096	2117.983	0.399
30	Goa	448	2700.791	0.306
31	Lakshadweep	192	3094.633	0.396
32	Kerala	4460	3014.732	0.431
33	Tamil Nadu	6647	2122.480	0.357
34	Pondicherry	576	3086.998	0.339
35	Andaman and Nicobar Is.	567	3937.967	0.347
	All India	101,717	1939.779	0.378

Data available for different states for 68th round (2011–2012)

Table 5 Number of households, average per capita consumption expenditure $E(x)$, Gini index for India

Filter	#Households	$E(x)$	Gini
ST	13,403	1601.763	0.338
SC	15,652	1507.782	0.335
OBC	39,721	1800.488	0.360
Other castes	32,938	2450.539	0.391
Hinduism	77,036	1935.365	0.384
Islam	13,274	1698.742	0.347
Christianity	6930	2223.477	0.357
Other religions	4477	2291.247	0.359
Rural	59,693	1525.498	0.322
Urban	42,024	2528.244	0.386

Data available religions, caste as well as urban–rural divide for 68th round (2011–2012)

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