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DISTRIBUTIONS OF MONEY IN MODEL MARKETS OF ECONOMY

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We study the distributions of money in a simple closed economic system for different types of monetary transactions. We know that for arbitrary and random sharing with locally conserving money transactions, the money distribution goes to the Gibb's distribution of statistical mechanics. We then consider the effects of savings, etc. and see how the distribution changes. We also propose a new model where the agents invest equal amounts of money in each transaction. We find that for short time-period, the money distribution obeys a power-law with an exponent very close to unity, and has an exponential tail; after a very long time, this distribution collapses and the entire amount of money goes to a tiny fraction of the population.

Keywords: Econophysics; simulation; market; power-law.

1. Introduction

Economics deals with the real life around us. The area of economics is not only restricted to the marketplace but also covers almost everything from the environment to family life! Economics is the study of how societies can use scarce resources efficiently to produce valuable commodities and distribute them among different people or economic agents.[1,2] Financial markets exhibit several properties that characterize complex systems. For financial markets, the governing rules are rather stable and the time evolution of the system can be continuously monitored. Recently, an increasing number of physicists have made attempts to analyze and model financial markets, and in general, economic systems.[3,4] The physics community took the first interest in financial and economic systems, when Majorana wrote a pioneering paper[5] on the essential analogy between statistical laws in physics and in the social sciences. This off-the-track outlook did not create much interest until recent times. In fact, prior to the 1990's, a very few professional physicists like Kadanoff[6] and Montroll,[7] took much interest in research in social or economic systems. Since 1990, physicists started turning to this interdisciplinary subject, and their research activity is complementary to the most traditional approaches of finance and mathematical finance.

It may be surprising to the students of physical sciences but the first use of the power-law distribution was made by an Italian social economist Pareto, a century ago, who investigated the wealth of individuals in a stable economy by modeling them using the distribution

$$y \sim x^{-\upsilon}\,,$$

where y is the number of people having income greater than or equal to x and υ is an exponent which he estimated to be 1.5.[8] Almost during the same time, the first formalization of a random walk was made by a French mathematician Bachelier in his doctoral thesis,[9] where he used the increments of Brownian motion to model "absolute" price changes. A major part of the recent efforts made by physicists has gone to investigating the nature of fluctuations and their distributions in the stock markets.[3] We believe that a thorough understanding of the statistical mechanics of the money market, especially the studying of the distribution functions, is essential. There have been some very interesting papers along this line.[10–14] Here, we make a very brief review of the earlier models and then propose a new variant where the agents invest equal amounts of money in each transaction. We find that for short time-period, the money distribution obeys a power-law with an exponent very close to unity, and has an exponential tail; after a very long time, this distribution collapses and the entire amount of money goes to a tiny fraction of the population.

2. Brief Review of Models and Distributions of Money

We consider a model of a closed economic system where the total amount of money M is conserved and the number of economic agents N is fixed. Each economic agent i, which may be an individual or a corporate entity, possesses money m_i. An economic agent can exchange money with any other agent through some trade, keeping the total amount of money of both the agents conserved. We assume that an agent's money must always be non-negative and therefore no debt is permitted.

2.1. *Random transactions*

Let an arbitrary pair of agents i and j get engaged in a trade so that their money m_i and m_j change by amounts Δm_i and Δm_j to become m_i' and m_j', where Δm_i is a random fraction of (m_i+m_j) and Δm_j is the rest of it, so that conservation of the total money in each trade is ensured. The money distribution goes to the equilibrium Gibb's distribution[15] of statistical mechanics: $P(m) = (1/T)\exp(-m/T)$ where "temperature" $T = M/N$, the average money per agent in the market, satisfying $P(m_i)P(m_j) = P(m_i + m_j)$.

Extensive numerical simulations show that this and various modifications of trade, like multi-agent transactions, etc., all lead to the robust Gibb's distribution (see Fig. 1), independent of the initial distribution the market starts with.[12] So, most of the agents end-up in this market with very little money.

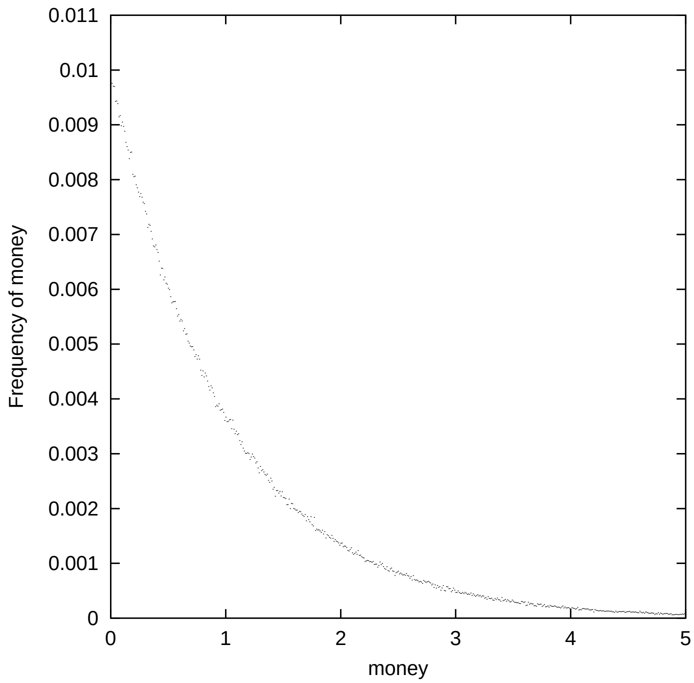

Fig. 1. Histogram of money, obtained from computer simulations made for agents $N = 1000$ and total money $M = 1000$. Note that in the figure, the money has been scaled by the average money in the market.

2.2. *Transactions with constant saving*

Let us introduce the concept of "saving", which is a very natural and important ingredient in economics, in our model. We assume that each economic agent saves a constant amount of money m_0 before trading. Let us now consider that an arbitrary pair of agents i and j get engaged in a trade so that their money m_i and m_j change by amounts Δm_i and Δm_j to become m'_i and m'_j; $\Delta m_i = \epsilon(m_i + m_j - 2m_0)$ and $\Delta m_j = (1 - \epsilon)(m_i + m_j - 2m_0)$, where ϵ is a random number between zero and unity, and $m'_i = m_0 + \Delta m_i$ and $m'_j = m_0 + \Delta m_j$ after the trade. In this case, the lower limit of money that an agent is allowed to possess actually changes from zero to m_0. Conservation of the total money in each trade is ensured, as earlier.

The probability distribution of money still remains as Gibb's distribution (see Fig. 2) but the money corresponding to maximum probability m_p, shifts from zero (or very little money) to m_0, and the "temperature" T changes. When $m_0 = 0$, we get back the case of Sec. 2.1. On the preceding page.

2.3. *Transactions with fractional saving (marginal propensity of saving)*

We now consider the case where each economic agent saves a fraction λ of its money m_i before trading.[11] This constant fraction of saving λ is called the "marginal propensity of saving" and is a very important quantity in economics.

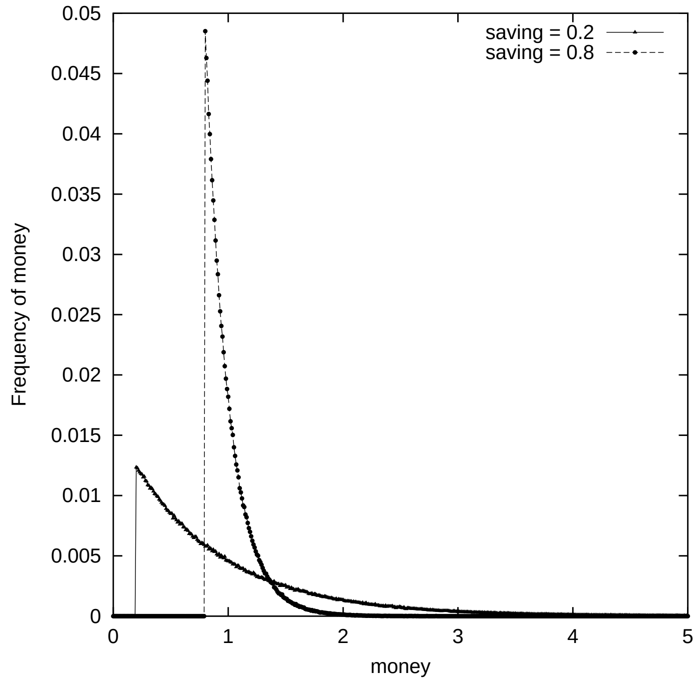

Fig. 2. Histogram of money for different saving amounts $m_0 = 0.2$ and $m_0 = 0.8$, obtained from computer simulations made for agents $N = 1000$ and total money $M = 1000$. Note that in the figure, the money has been scaled by the average money in the market.

Here, we choose randomly two agents i and j having money m_i and m_j, respectively. Then $\Delta m_i = \epsilon(1-\lambda)(m_i + m_j)$ and $\Delta m_j = (1-\epsilon)(1-\lambda)(m_i + m_j)$, where ϵ is a random number between zero and unity. Then $m'_i = \lambda m_i + \Delta m_i$ and $m'_j = \lambda m_j + \Delta m_j$ after the trade. Conservation of the total money in each trade is ensured, as earlier.

The results for the equilibrium distribution $P(m)$ are shown in Fig. 3, for some values of λ. The real money exchanged randomly in any trade is less than the total money, because of the saving by each agent. This destroys the multiplicative property of the distribution $P(m)$ (seen earlier for $\lambda = 0$) and $P(m)$ changes from the Gibb's form to the asymmetric Gaussian-like form as soon as a finite λ is introduced. The $\lambda = 0$ case, the same as in Sec. 2.1, on the page before, was practically a random-noise dominated one and therefore effectively a noninteracting market. Introduction of a finite amount of saving ($\lambda \neq 0$), dictated by individual self-interest, immediately makes the money dynamics cooperative and the global ordering (in the distribution) is achieved.

An important feature of this humped distribution $P(m)$ at any nonvanishing λ is the variation of the most probable money $m_p(\lambda)$ (where $P(m)$ becomes maximum) of the agents.[11] We have, $m_p = 0$ for $\lambda = 0$ (Gibb's distribution) and most of the economic agents in the market end-up losing most of their money. However, even with the pure self-interest of each agent for saving a factor λ of *its* own money in any trade, a global feature emerges: the entire market ends-up with a most-probable

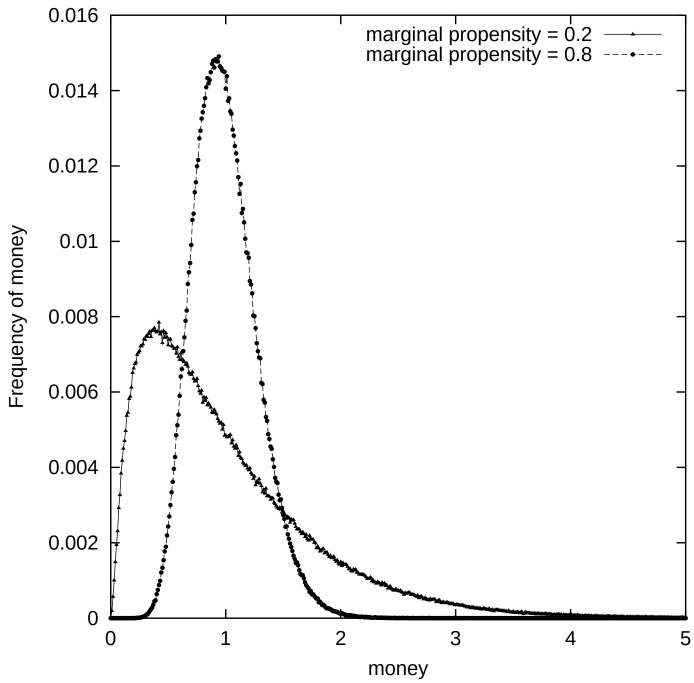

Fig. 3. Histogram of money for different saving propensity factor $\lambda = 0.2$ and $\lambda = 0.8$, obtained from computer simulations made for agents $N = 1000$ and total money $M = 1000$. Note that in the figure, the money has been scaled by the average money in the market.

money $m_p(\lambda)$. This $m_p(\lambda)$ shifts in an interesting manner from $m_p = 0$ (for $\lambda = 0$) to $m_p \to T$ (for $\lambda \to 1$). The half-width Δm_p and the peak height P_{m_p} of the equilibrium distribution scales practically as $(1-\lambda)^{1/2}$ and $(1-\lambda)^{-1/2}$, respectively. We also note that each individual's money m_i fluctuates randomly. Since the total money is conserved, $\langle m_i \rangle$ remains constant $(= T)$ here, while Δm_i goes down with λ as $(1 - \lambda)$. This is because at any time the agents keep a fixed fraction of their individual money and receive a random fraction of the money traded that is proportional to $(1 - \lambda)$.

3. Model and Simulation Results

We now introduce a new model where we assume that both the economic agents invest the same amount of money $m_{\min}$, the minimum money between the agents. We choose randomly two agents i and j having money m_i and m_j, respectively. Thus $2m_{\min}$ is the real money which is available in the market for random sharing. Then $\Delta m_i = \epsilon 2m_{\min}$ and $\Delta m_j = (1 - \epsilon)2m_{\min}$, where ϵ is a random number between zero and unity. Then $m'_i = (m_i - m_{\min}) + \Delta m_i$ and $m'_j = (m_j - m_{\min}) + \Delta m_j$ after the trade. Conservation of the total money in each trade is ensured, as earlier. Note that we may rewrite the amounts of money after trade as $m'_i = m_i + \alpha m_{\min}$ and $m'_j = m_j - \alpha m_{\min}$, where $\alpha(= 2\epsilon - 1)$ is a random fraction whose absolute value is less than unity, i.e. $-1 < \alpha < 1$ and $m_{\min}$ is either m_i or m_j, whichever is less. We consider one transaction between any arbitrary pair of agents as a unit of "time", t. We made computer simulations for agents $N = 1000$ and the total money in the market $M = 1000$ and then took averages over 5000 different configurations. We studied the money distribution $P(m)$ at different times.

The money distribution $P(m)$ for time $t = 10\,000$ is shown in Fig. 4. We find that the distribution obeys a power-law: $P(m) \sim m^{-\upsilon}$, where υ is an exponent, and has a tail which falls off exponentially ($\sim \exp(-\alpha m)$). The numerically fitted curves (indicated by the solid line and the dashed curve in the figure) give the following exponents: $\upsilon = 0.9 \pm 0.01$, which is very close to unity, and $\alpha = 0.25 \pm 0.02$. The errors are obtained by eye-estimation.

We note that once an agent loses all its money, it is unable to trade any more because $m_{\min}$ becomes zero and no other agent will invest money for trade with this agent. Thus, a trader is effectively driven out of the market once it loses all its money. In this way, after an infinite number of transactions have taken place, one would expect that only one trader survives in the market with the entire amount of money and the rest of the traders have zero money. In our numerical simulations, we found that for $t = 15\,000\,000$, more than 99% of the traders have zero money and the rest have the entire money of the market. This can be prevented, for example,

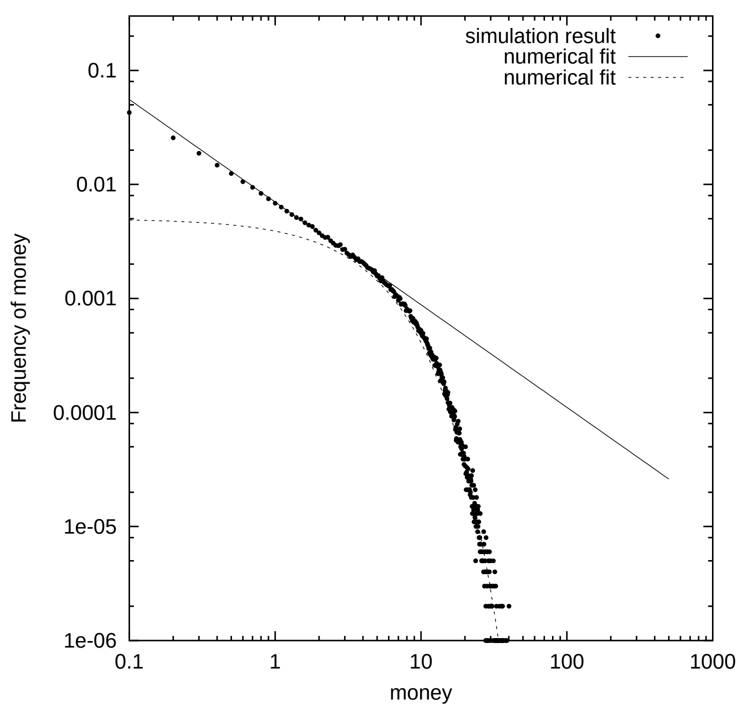

Fig. 4. Histogram of money plotted in the double logarithmic scale, obtained from computer simulations made for agents $N = 1000$ and total money $M = 1000$, at time $t = 10\,000$. The solid line is the numerically fitted line with slope $\upsilon = 0.9$ and the dashed curve is the numerically fitted exponential curve with an exponent $\alpha = 0.25$. Note that in the figure, the money has been scaled by the average money in the market.

by government intervention via taxes, but we do not consider any such thing in our model.

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References

1. P. A. Samuelson, *Economics*, 16th edition (McGraw-Hill Inc., Auckland, 1998).
2. J. M. Keynes, *The General Theory of Employment, Interest and Money* (The Royal Economic Society, Macmillan Press, London, 1973).
3. R. N. Mantegna and H. E. Stanley, *An Introduction to Econophysics* (Cambridge University Press, New York, 2000); S. M. de Oliveira, P. M. C. de Oliveira, and D. Stauffer, *Evolution, Money, War and Computers* (B. G. Teubner, Stuttgart-Leipzig, 1999).
4. J. P. Bouchaud and M. Potters, *Theory of Financial Risk* (Cambridge University Press, Cambridge, 2000); H. Levy, M. Levy, and S. Solomon, *Microscopic Simulation of Financial Markets* (Academic Press, New York, 2000).
5. E. Majorana, *Scientia* **36**, 58 (1942).
6. L. P. Kadanoff, *Simulation* **16**, 261 (1971).
7. E. W. Montroll and W. W. Badger, *Introduction to Quantitative Aspects of Social Phenomena* (Gordon and Breach, New York, 1974).
8. V. Pareto, *Cours d'Economie Politique* (Lausanne and Paris, 1897).
9. L. Bachelier, *Annales Scientifiques de l'Ecole Normale Superieure* **III-7**, 21 (1900).
10. M. Levy and S. Solomon, *Physica A* **242**, 90 (1997).
11. A. Chakraborti and B. K. Chakrabarti, *Eur. Phys. J. B* **17**, 167 (2000).
12. A. Dragulescu and V. M. Yakovenko, *Eur. Phys. J. B* **17**, 723 (2000); A. Dragulescu and V. M. Yakovenko, *Eur. Phys. J. B* **20**, 585 (2001); A. Dragulescu and V. M. Yakovenko, *Physica A* **299**, 213 (2001).
13. A. Chakraborti, S. Pradhan, and B. K. Chakrabarti, *Physica A* **297**, 253 (2001).
14. W. Souma, *preprint available at cond-mat/0202388* (2002).
15. F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, Singapore, 1985); L. D. Landau and E. M. Lifshitz, *Statistical Physics*, 3rd edition (Part I) (Butterworth-Heinemann, Oxford, 1998).